JUNIOR MATHEMATICS

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JUNIOR MATHEMATICS

BOOK THREE

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JUNIOR MATHEMATICS

BOOK THREE

BY

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PREFACE

This volume is the third of a series of three books in junior high school mathematics. It is hoped that the junior high school with a continuous three-year course in mathematics will be able to accomplish more than has been possible with the traditional organization of two years of elementary school mathematics and one year of high school algebra. The complete course of Books I, II, and III contains the maximum of instructional material which may be taught in grades seven, eight, and nine. The subject matter is planned and arranged to meet the varying needs of different schools, classes, and individuals. Omissions of material not essential in the future work of the course are indicated in the text.

Thus pupils who intend to continue the study of mathematics are able to master the requirements of algebra of the senior high school in the three years of the junior high school. In addition they acquire a large amount of intuitive geometry. They are therefore thoroughly prepared for the study of demonstrative geometry and trigonometry.

Schools wishing to minimize the emphasis on mathematics may complete only Books I and II, and possibly some selected topics from Book III. When entering the high school their pupils will be prepared to follow the regular senior high school course beginning with the mathematics of the tenth grade.

In Book III the plan of Books I and II is carried forward, *i.e.* arithmetic, algebra, and geometry are taught together.

In Book I and in the first part of Book II intuitive geometry is the basis of the work. Algebra is brought in wherever it is needed or is helpful. In the second part of Book II the emphasis shifts from geometry to algebra. Geometry is now used to illustrate and make concrete the abstract concepts, principles, and processes of algebra. This continues through Book III.

The author expresses his gratitude to Director Charles H. Judd, Professor H. C. Morrison, and Professor W. C. Reavis for their encouragement, assistance, and inspiration during the period of experimentation, and to Charles A. Stone, James W. Hoge, and J. S. Georges, instructors of mathematics in the University High School, for their helpful criticisms and for their coöperation while trying out the materials in manuscript form in their classes.

Through a subsidy from the Commonwealth Fund the author was enabled to visit the leading junior high schools of the country. The study of these schools has been exceedingly helpful in selecting and organizing the material used in this course.

E. R. Breslich.

CONTENTS

CHAPTER I

A STUDY OF POLYNOMIALS OF THE FIRST DEGREE

Relationships in Mathematics

1. Relationships between numbers in everyday affairs. Relationships between numbers play an important rôle in the affairs of life. For example, a man's taxes depend on the value of his property or on the amount of his income; the interest received on an investment is related to the amount invested and to the rate of interest; a life-insurance premium depends upon the amount of insurance carried; the cost of a railroad ticket depends upon the distance to be traveled.

Problems involving relationships have to be solved by people in all occupations. The boy sending a parcel

computes the charge from the weight of the parcel and the rate per ounce. The farmer building a fence finds the cost from the distance around the field and the price of fencing material per yard. The cook preparing a meal determines the quantity of food from the number of people to be served. Mathematics trains us in solving problems that involve relationships.

2. Relationships in geometry studied by means of equations and formulas. You are familiar with many of these relations. You know that when two angles of a triangle are given, the size of the third is definitely fixed, because it depends upon the sum of the other two angles. The size of an equilateral triangle depends upon the length of the side. The area of a rectangle depends upon the lengths of base and altitude. The volume of a cube depends upon the length of the edge. You have studied these and other relationships in previous courses. Many of them were expressed in terms of algebraic equations or formulas. Thus, the circumference of a circle was found by means of the formula $c=2\pi r$. The number c depends on r for its value. If you let r change in value, c changes also in value, the nature of the change being determined by the relation $c=2\pi r$. This relation must always be satisfied by r and the corresponding value of c. The literal numbers r and c are said to *vary* (change) and are called *variables*, the number π is a *constant* (remaining the same), and c is said *to depend upon* r. The variable c is also called a *function* of the variable r, which means that for every value of r the equation $c=2\pi r$ determines a corresponding value of c.

Similarly, the equation $A = \pi r^2$ shows that the variable area A of a circle depends upon the variable length r of the radius. Hence, the numbers r and A are variables, π being a constant.

The equation $A = \frac{a^2}{4}\sqrt{3}$ shows that the area of an equilateral triangle depends upon the length of the side. The variables are a and A. The constant is $\frac{\sqrt{3}}{4}$.

The equation $V = \frac{1}{3}bh$ shows that the volume of a cone depends upon the base and altitude. The constant is $\frac{1}{3}$; the variables are b, h, and V. Here the value of V depends upon *two* variables, b and h.

The sides a, b, and c, of a right triangle are related by means of the equation $a^2 + b^2 = c^2$. If two sides are selected the third can be determined by solving the equation.

3. What is meant by a function. The variable A in the relation $A = 5h$ is said to be a *function* of h. This means that A depends upon h for its value, and that to every value of h there corresponds a value of A. When one variable is so related to another variable that to every value of the second the first has one or several corresponding values, the first variable is a **function** of the second. Thus, the word *function*, when used in mathematics, means relationship, dependence, and correspondence of several variables.

Note that one variable may be a function of several other variables. For example, in $c^2 = a^2 + b^2$ the variable c is a function of a and b.

Exercises

1. Give examples of relationships which occur in the following occupations: farming, cooking, selling, buying, building, banking, engineering.

2. Complete the following statements:

a. If a train travels at a uniform rate, the distance is a function of ——.

b. The perimeter of an equilateral triangle is a function of ——.

c. The amount of interest received from an investment is a function of ——.

d. The premium of a life-insurance policy is a function of ——.

e. The amount of electricity used in a house is a function of ——.

f. The temperature at a given place is a function of ——.

g. The amount of work a man does depends upon the number of ——.

h. The number of articles that can be bought for $3 is a function of ——.

i. The weight of an iron rod depends upon ——.

3. In each of the following equations tell what the literal numbers stand for. Name the constants and the variables.

a. $d=20t$

b. $C=\frac{5}{9}(F-32)$

c. $A=180-(B+C)$

d. $p=\dfrac{5b}{100}$

e. $A=a^2$

f. $v=\frac{4}{3}\pi r^3$

g. $s=4\pi r^2$

h. $s=\frac{1}{2}gt^2$

Variation

4. An important relation in algebra. Sometimes it is possible to give an exact algebraic expression for the relation between two variables. Thus, we know that the circumference of a circle is given by the

equation $c = 2\pi r$, that the distance traveled by a train moving at a uniform rate of 40 miles an hour is given by $d = 40t$, that the interest on \$2000 invested at a rate of 6% is given by the formula $i = 120t$. If you examine these equations you will see that all three are of the *same form*. Furthermore, you will notice that each contains two variables; that as one changes, the other changes also; and that in each case the *ratio of the variables remains constant*. The three relations above may be said to be of the form $y = cx$, where c is the constant ratio $\frac{y}{x}$. When two variables are related this way, one is said to *vary directly* as the other.

François Viète (Vieta) (1540–1603), a French mathematician, studied mathematics as a leisure occupation because it gave him great pleasure and satisfaction. Although he did not devote all of his time to the study, he achieved the fame of being the greatest of French mathematicians of the 16th century.

His interest was chiefly in algebra and trigonometry. He was one of the first writers to use letters to denote numbers. He used vowels to denote unknowns and consonants to represent known numbers. Exponents, as we now teach them, were not known at his time, but he introduced a better notation than the one in use. He denoted the square of an unknown number by Aq, the cube by Ac, and the fourth power by Aqq.

We may now summarize the discussion as follows:

The statement, "a variable y varies directly as a variable x" *is expressed algebraically by the relation* $y = cx$. Conversely, *if* $y = cx$, y *is said to vary directly*

as x. The literal number c is the *constant of variation*, and the equation $y=cx$ is the *law of variation.*

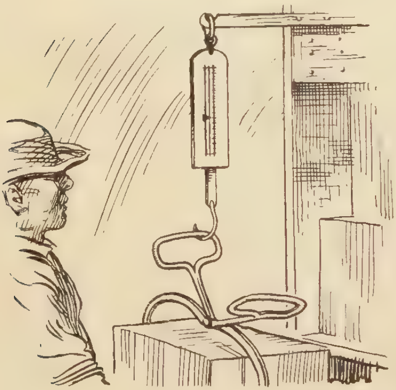
Exercises

Express the following relations algebraically:

1. The cost of sugar varies directly as the weight.

Solution: Using the first letter of the word to denote the variable, we have $C=cw$.

2. The weight of a liquid varies directly as the volume.



3. The weight of a beam varies as the length.

4. The diagonal of a cube varies directly as the edge.

5. The distance sound travels varies directly as the time.

6. The distance a spring is stretched by a weight varies as the weight.

7. The distance (in feet) an object falls from rest varies directly as the square of the time (in seconds).

8. The pressure in pounds per square inch of a column of water varies directly as the height of the column.

9. The weight of a sphere varies as the cube of the radius.

10. The time required by a pendulum to make one vibration varies directly as the square root of its length.

11. The area of an equilateral triangle varies as the square of a side.

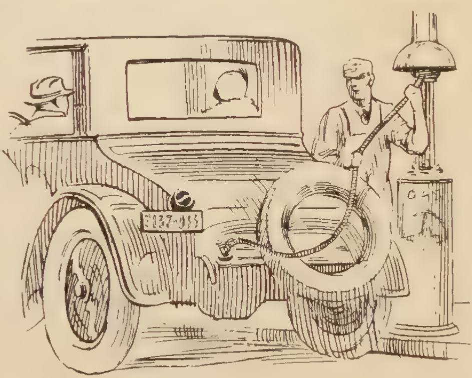
12. The volume of a cylinder varies as the square of the radius of the base, if the altitude remains constant.

5. How to determine the constant of variation. You have seen that the statement, "The distance traveled by a train varies directly as the time" may be expressed briefly by means of the equation $d=ct$. To determine the constant c, you must know one pair of corresponding values of t and d. Thus, if it is known that the train traveled 70 miles in 2 hours, you have $d=70$ if $t=2$. A substitution of these values of d and t in the equation $d=ct$ gives $70=c\times 2$. It follows that $c=35$.

In each of the exercises below determine the constant of variation.

Exercises

1. The amount of gasoline used in driving an automobile varies directly as the number of miles traveled by it. If an automobile uses 6 gallons to travel 72 miles, what is the constant of variation?



Solution: Since the amount of gasoline varies directly as the distance, it follows that $a=cd$. Using subscripts to denote corresponding values of a and d, we have $a_1=6$ if $d_1=72$.

Hence, $6=c\times 72$

$$\therefore c=\tfrac{1}{12}$$

$$\therefore a=\tfrac{1}{12}d.$$

The constant of variation in this problem denotes the amount of gasoline used per mile.

2. When a spring is stretched by a weight w, the amount of the stretch s varies as the weight. When a weight of 18 pounds stretches the spring 3 inches, find the constant of variation and

state the general law of variation in symbols. What is the meaning of the constant of variation?

3. The cost C of a certain grade of butter varies as the number of pounds n. If 6 pounds of butter cost \$2.64, find the constant and state the law of variation in symbols. State the meaning of the constant of variation.

4. If the altitude of a rectangle is constant, the area varies as the base. If the area is 20 when the base is 4, write the law of variation. What is the meaning of the constant?

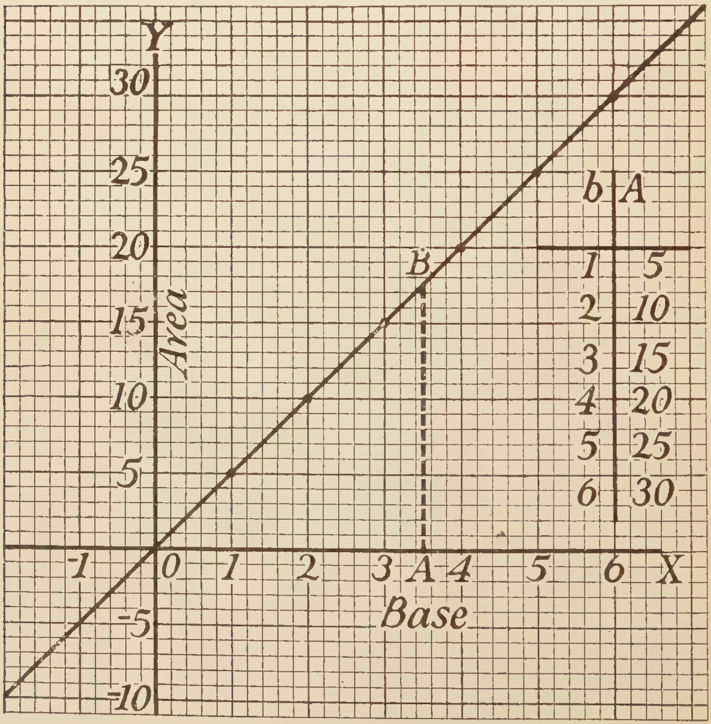


Fig. 1

6. Solving problems in variation graphically. You have seen that the law of variation for Exercise 4 (§5) is $A=5b$. Let it be required to find the area when $b=3.5$.

Solution: 1. By substituting values for b, find the corresponding values of A as given in the table of Fig. 1. Thus, if $b_1=1$, then $A_1=5$; if $b_2=2$, $A_2=10$; etc.

2. Draw the reference axes OX and OY and plot the pairs of numbers given in the table. For example, to plot the pair (1, 5) pass from the origin O one unit to the right and then place a mark at a distance of 5 units upward from that point.

3. Draw the line passing through the marked points. This is the required graph of the equation $A=5b$.

4. To find the area when $b=3.5$, pass from the origin O to the right a distance equal to 3.5 to point A. Then pass upward to the graph at B and read off the length of the vertical line segment AB.

Exercises

1. From the graph (Fig. 1) find values of A corresponding to the following values of b: 2.5; 5.2; 1.8; 4.4.

2. The diagonal of a cube varies as the edge. If the diagonal is approximately 8.5 inches when the edge is 5 inches, find the constant of variation. State the law of variation and represent it graphically. From the graph find the approximate lengths of diagonals of cubes having the following edges: 2; 4; 4.6; 3.2.

Suggestion: Use the method explained in §6.

3. The weight of steel wire varies as the length. If 10 feet of wire weigh 2 pounds, find graphically the weights of the same kind of wire of the following lengths: 2; 8; 4.5; 5.

4. From the graph of Exercise 3 find the lengths of wire weighing 1 pound; 3 pounds; 4.5 pounds.

5. The speed (velocity) of a falling object varies as the time. If an object has fallen 3 seconds, it has a speed of 48 feet a second. Find graphically the speed attained by an object when it has fallen the following numbers of seconds: 2; 6; 4; 3.5.

6. From the graph of Exercise 5 find the time it takes a falling object to attain a speed of 64 feet a second; 40 feet a second.

7. Represent graphically the equation $y=cx$ when $c=3$.

7. Solving problems in variation by proportions. A convenient method of solving problems in variation is found in the use of proportions. The example below explains the method.

In Exercise 2 (§6) it was stated that the diagonal of a cube varies as the edge and that the diagonal is about 8.5 inches when the edge is 5 inches. It was required to find the diagonal of a cube whose edge is 2 inches. We may solve the problem as follows:

1. State the law of variation. This is $d=ce$.

2. You know that this equation is true for any pair of corresponding values of d and e, for example for the pairs (d_1, e_1) and (d_2, e_2).

Hence, $d_1=ce_1$

and $d_2=ce_2$

3. Dividing the first equation by the second you have

$$\frac{d_1}{d_2}=\frac{e_1}{e_2}$$

4. Since in the problem above $d_1=8.5$, $e_1=5$, and $e_2=2$, you may substitute these values in the proportion above. This gives

$$\frac{8.5}{d_2}=\frac{5}{2}.$$

5. Hence $5d_2 = (8.5) \times 2$
and $d_2 = 3.4$

In solving problems in variation by proportions you need to use only steps 1, 3, 4, and 5.

Exercises

Solve the following problems using proportions:

1. The amount of gasoline used in driving an automobile varies directly as the number of miles traveled by it. If an automobile running at a uniform rate in the country uses 5 gallons of gasoline to travel a distance of 52 miles, how much gasoline will be needed for a trip of 90 miles?

Solution: $a = cd$

$$\therefore \frac{a_1}{a_2} = \frac{d_1}{d_2}$$

Let $a_1 = 5$, $d_1 = 52$, and $d_2 = 90$.

Then by substitution you have

$$\frac{5}{a_2} = \frac{\overset{26}{\cancel{52}}}{\underset{45}{\cancel{90}}}$$

$\therefore 26a_2 = 225$

$\therefore \quad a_2 = 8.7$ approximately.

2. The mass of an object (amount of matter in it) of uniform material varies as the volume. If the mass of an aluminum object is 17 when the volume is 6, find the mass of an aluminum object made of the same material whose volume is 9.35.

3. The velocity of an object falling from rest varies as the time. If a particle falling 6 seconds has a speed of 96 feet a second, what was its speed in 4 seconds?

4. The diagonal of a cube varies as the edge. Since the edge is about 10 when the diagonal is 17, find the diagonal when the edge is 8.

5. The distance which an object falls varies as the square of the time. If an object falls 144.7 feet in 3 seconds, how far will it fall in 5 seconds?

Suggestion: $\frac{d_1}{d_2}=\frac{t_1^2}{t_2^2}$

6. The area of a circle varies as the square of the radius. The area of a circle is approximately 12.56 when the radius is 2. Find the area when the radius is 5.

7. The time required by a pendulum to make a vibration varies as the square root of the length. If a pendulum 100 centimeters long makes one vibration in 1 second, what is the time of one vibration of a pendulum 64 centimeters long?

8. The voltage of a dynamo varies directly as the speed. If the speed is 225 revolutions a minute the voltage is 183.8. Find the voltage if the speed is 800 revolutions a minute.

9. The elongation of a spring varies directly as the weight. The elongation is 8 when the weight is 100. Find the elongation when the weight is 70.

10. The volume of a gas under constant pressure is directly proportional to the absolute temperature. The volume is 300 when the temperature is 150. Find the volume when the temperature is 173.

11. The number of tons of coal used per hour in a locomotive varies as the square of the number of miles per hour. If a locomotive consumes 4 tons of coal at a rate of 40 miles, what is the speed obtained when 5 tons are used per hour?

12. The weight of a liquid varies directly as the volume. If 10 cubic feet of water weigh 625 pounds, find the weight of 37 cubic feet.

13. The work done by a machine varies as the number of hours. Working 6 hours a machine does the work of 120,800 foot-tons. How much work will it do in 20 minutes?

A Number Series Formed According to an Important Law

8. Problems leading to arithmetical progressions. In traveling through the Cumberland Mountains John's father started his car downhill in second gear at the rate of 7 feet a second (about 5 miles an hour) and turned off the power. The car's speed increased uniformly at the rate of 2 feet a second. How fast was it going 12 seconds later?

To solve this problem write the distances traveled during the 12 seconds: 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29.

The answer is 29 feet per second, which is almost 11 miles per hour. By examining the series 7, 9, 11, etc., notice that it is formed according to a definite law, each term being formed from the preceding term by adding to it the number 2. Such a series of numbers is called an *arithmetical progression*.

Find the sum $7+9+11+ \ldots +29$, to find the distance traveled in the 12 seconds.

Having completed the eighth grade, Henry secured a job for the summer in a store on 63d Street. He had asked for $14 a week. The owner of the store was not willing to pay that amount, but agreed to pay him $12 the first week and a 50 cents' increase each week for 10 weeks. Both felt that this was a satisfactory arrangement. How much would Henry have earned in 10 weeks at the rate of $14 a week? How much did he actually earn? What was his pay for the tenth week?

To answer the questions, write the arithmetical progression 12.00, 12.50, 13.00, . . . to 10 terms, and from it find the tenth term and the sum.

9. A formula for finding a required term in an arithmetical progression. Algebra enables you to solve the two preceding problems by means of formulas. This is a quicker and easier way than the direct arithmetical method used above. The formulas are found as follows:

Number of Term	Term
1	a
2	$a+d$
3	$a+2d$
4	$a+3d$
5	$a+4d$
6	$a+5d$
etc.	etc.

Fig. 2

Let a be the first term in the progression and let d be the number which is added to each term to give the following term. Then the terms of the progression may be arranged in the form of a table (Fig. 2). By examining the table you will see that the coefficient of d is always 1 less than the number of the term. You may therefore write at once

the 10th term: $a+9d$; the 15th term: $a+14d$; the 18th term: $a+17d$; etc.

In general, the nth term (any term) is $a+(n-1)d$.

Calling the nth term l, you have the formula

$$l=a+(n-1)d.$$

The formula shows that l is a function of a, n, and d.

Returning now to the first of the two problems above, show that $a=7$, $d=2$, $n=12$.

The substitution of these values in the formula $l=a+(n-1)d$ gives $l=7+(11\times 2)=7+22=29$.

In the second problem $a=12$, $d=\frac{1}{2}$, $n=10$.

$$\therefore l=12+(10-1)\times\tfrac{1}{2}$$
$$\text{or } l=12+\tfrac{9}{2}=16\tfrac{1}{2}.$$

Hence, Henry earned \$16.50 during the tenth week.

The formula for finding the sum will be developed later. Use the formula $l=a+(n-1)d$ in the following exercises:

Exercises

1. Henry is learning to read Latin. At the beginning of the month he can read twenty lines in one lesson. His lessons become five lines longer each week. In ten weeks how many lines will he read?

2. A man begins work in an office at a salary of \$2200 a year. His salary is increased \$125 a year. How much is he earning during the ninth year?

3. John's brother increases his savings from year to year. He saves \$100 the first year, \$175 the second, \$250 the third, etc. How much does he save during his fifteenth year?

4. A bobsled going down a hill travels 8 feet the first second and increases in speed by 16 feet each succeeding second. If the distance from the top to the bottom of the hill is covered in 6 seconds, how fast was the sled going when it reached the foot of the hill?

5. Ruth's parents have bought a house and lot. They paid down a sum of \$5000 and agreed to pay \$500 at the end of the first year, \$550 at the end of the second, etc., for 10 years. How large will be the last payment?

6. Choosing any convenient length for a and another for d, make a graph of the formula $l=a+(n-1)d$.

Suggestions: Let $a=\frac{1}{2}$ cm., $d=1$ cm., and let n take the values 0, 1, 2, 3, . . .

Use the method of §6.

10. A formula for finding the sum of the terms of an arithmetical progression. That the sum s of the series of numbers $7+9+11+13+15+17+19+21+23+25+27+29$ may be found in a simple manner will be seen from the following:

Solution: 1. Write the sum from the first to the last term as shown below.

2. Write the sum in the reverse order.

3. Add the two sums.

$$
\begin{array}{rl}
\text{Thus let} & s = 7+9+11+\ldots+25+27+29. \\
\text{Then let} & s = 29+27+25+\ldots+11+9+7. \\
\hline
\text{Adding, you have} & 2s = 36+36+36+\ldots+36+36+36.
\end{array}
$$

Since there are 12 terms in the right member of the last equation, it follows that

$$2s = 12 \times 36$$
$$\text{and } s = 216.$$

The same method is used to derive the formula for finding the sum:

Let

$$s = a + (a+d) + (a+2d) + \ldots + (l-d) + l.$$

Then let

$$s = l + (l-d) + (l-2d) + \ldots + (a+d) + a.$$

Adding,

$$2s = (a+l) + (a+l) + (a+l) + \ldots + (a+l) + (a+l)$$

$\therefore 2s = n(a+l)$, if there are n terms in the series.

$$\therefore s = \frac{n}{2}(a+l).$$

The sum $7+9+11+\ldots+29$ may now be found easily by using this formula:

Solution: The number of terms is 12. $\therefore n = 12$
The first term is 7. $\therefore a = 7$
The last term is 29. $\therefore l = 29.$

Substituting these values in the formula, you have

$$l=\frac{\overset{6}{\cancel{12}}}{\cancel{2}}(7+29)$$

$$l=6(7+29)$$

$$l=216.$$

In the second illustration given in §8 you have

$$a=12$$

$$n=10$$

$$d=.50=\tfrac{1}{2}.$$

Using the formula $l=a+(n-1)d$,

you have $l=12+(9\times\frac{1}{2})$

or $l=16\frac{1}{2}$.

Using the formula $s=\frac{n}{2}(a+l)$

you have $s=\frac{10}{2}(12+16\frac{1}{2})$

or $s=5\times28\frac{1}{2}=142.50$.

Exercises

1. A ball is dropped and falls 16 feet in the first second, 48 in the next, etc., *i.e.*, in each second it falls 32 feet more than in the preceding. What distance does the ball fall in 10 seconds?

Solution: The series of distance is

$$s=16+48+80+\ \dots$$

$$d=32$$

$$n=10.$$

Find l from the formula $l=a+(n-1)d$.

Then find s from the formula $s=\frac{n}{2}(a+l)$.

2. How many strokes does a clock strike in 12 hours, if it strikes the hours but not the half hours?

3. Find the sum of all odd integers (whole numbers) between 0 and 100.

4. A car sliding down-grade moves 6 inches the first second, 14 the second, 22 the third, etc. How far does it move in 18 seconds?

5. A man buying a lot makes a first payment of $50 and agrees to pay the remainder on the installment plan. His first monthly payments are $10.00, $10.25, $10.50, etc. How much has he paid in one year?

6. By means of the formula find the 14th term of 2, 4, 6, 8, . . .

7. Find the 19th term of 4, $4\frac{1}{2}$, 5, $5\frac{1}{2}$, . . .

8. Find the sum of all integers which are divisible by 3 and lie between 0 and 190.

9. In a potato race 20 potatoes are placed in a straight line and 1 yard apart from each other. The first potato is 4 yards from the basket. What is the total distance a boy must run to carry the potatoes to the basket, taking but one at a time?

10. A man begins work in an office at a salary of $2000 a year. Each year he receives a salary increase of $100. What will he earn in the 12th year, and what will be the total amount earned in 12 years?

11. Henry's father buys a $700 piano, paying down $100 and promising to pay $30 a month plus interest at 7%. How much interest will he have to pay in the first twelve months? What will be the amount paid during the first year?

12. The speed of a falling object is 64 feet per second at the end of the second second, a second later it is 96 feet, and it continues to increase uniformly. How many feet does it fall during the fifth second?

13. If a baseball is dropped from the top of the Washington Monument (555 feet), how long does it take it to reach the ground?

14. A sum of $500 is invested at 6% simple interest. What will be the amount of principal and interest after 10 years?

15. Four numbers are in arithmetical progression. If the sum of the first two is 12 and that of the remaining two is -20. what are the four numbers?

16. In the progression 3, 6, 9, 12, . . . to 15 terms find the 15th term and the sum of the 15 terms.

17. Show that the sum of the first n terms of the arithmetical progression $1+3+5$ . . . is n^2.

Linear Polynomials

11. What is meant by a linear function. Monomials and binomials like $3x$, $\frac{1}{2}x$, $2\pi r$, $30t$, $2y-6$, $\frac{9}{5}C+32$, v_0+gt are called *linear functions*. In a linear function the variable occurs in the first degree only. If you write $3x=3x+0$, it is seen that the binomial $3x+0$ is of the form $ax+b$ where $a=3$ and $b=0$. You may write $2\pi r=2\pi r+0$, which is of the form $ar+b$ where $a=2$ and $b=0$. The binomial $2y-6=2y+(-6)$, is of the form $ay+b$ where $a=2$, $b=-6$. Similarly, it may be shown that each of the functions above is a particular case of the general form $ax+b$.

Exercises

1. Show as above that the functions $\frac{1}{2}x$, $30t$, $\frac{9}{5}C+32$, and v_0+gt are of the form $ax+b$ and therefore linear functions.

2. Make up several linear functions. Show that each is of the form $ax+b$.

12. Finding values of a linear function. If you let x take the value 1, then

$$2x-5=2(1)-5=2-5=-3.$$

When $x=2$, then $2x-5=2(2)-5=4-5=-1.$

Thus the value of a linear function, as $2x-5$, depends upon the value of x, and is found by substituting a value of x in place of x. A convenient abbreviated form for "function of x" is $f(x)$. This symbol is read *function of x* or more briefly *f of x*. Note carefully that

the f in the symbol is not a number, but an abbreviation for the word "function." Read the following: $f(y)$, $f(r)$, $f(t)$.

If $f(x)$ denotes a function of x, then $f(2)$ denotes the *value* of that function when $x=2$.

For example, let $f(x)$ denote the binomial $4x-3$.

Then show that
$$\begin{aligned}
f(1) &= 4(1)-3 = 4-3 = 1\\
f(2) &= 4(2)-3 = 8-3 = 5\\
f(3) &= 4(3)-3 = 12-3 = 9\\
f(-1) &= 4(-1)-3 = -4-3 = -7\\
f(0) &= 4(0)-3 = 0-3 = -3.
\end{aligned}$$

Exercises

1. If $f(x)=30x$, find $f(1)$, $f(2)$, $f(-5)$, $f(0)$.

2. If $f(r)=2r$, find $f(0)$, $f(1)$, $f(2)$, $f(-1)$, $f(-6)$.

3. If $f(y)=3y-10$, find $f(0)$, $f(1)$, $f(2)$, $f(-8)$.

4. If $f(x)=2x-3$, find $f(4)$, $f(0)$, $f(-4)$, $f(2)$.

5. If $f(x)=4x-8$, make a table of values of $f(x)$ for $x=0, 1, 2, \ldots 10$.

13. Making a graph of the linear function $ax+b$. It can be shown by advanced mathematical methods that the graph of a linear function is always a straight line.

Construct the graph of each of the functions below and discuss each as shown in Exercise 1:

Exercises

1. $f(x)=2x-1$.

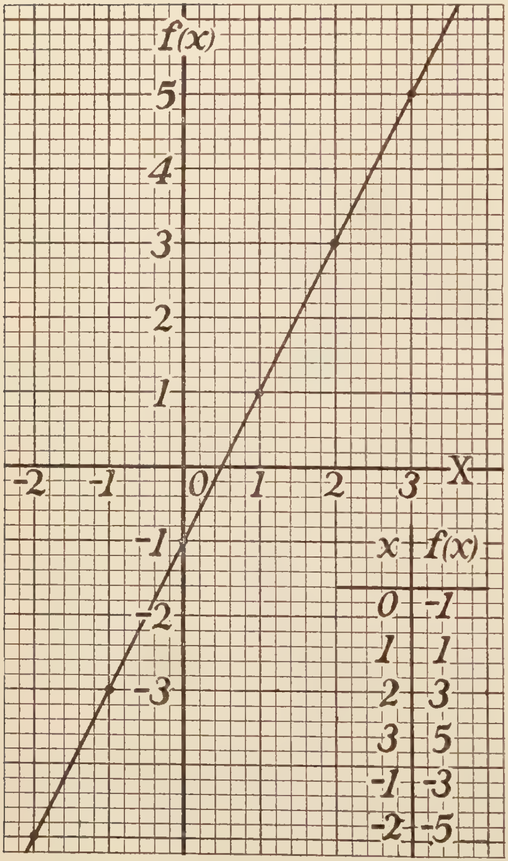


Fig. 3

Solution: Make a table of corresponding values of x and $f(x)$ (Fig. 3).

Draw the reference axes and select convenient units for plotting the number pairs in the table.

Plot the points corresponding to the number pairs in the table.

Draw the straight line passing through these points.

Discussion: When $x=0$, $f(x)=-1$.

As x varies, increasing from 0 to $\frac{1}{2}$, $f(x)$ varies from -1 to 0.

As x continues to increase, $f(x)$ also increases.

As x varies from 0 to the negative side, $f(x)$ varies, always remaining negative.

2. $f(x)=x$.

3. $f(x)=2x-3$.

4. $f(x)=3(x-1)$.

5. $f(r)=2\pi r$.

6. $f(C)=\frac{9}{5}C+32$.
7. $f(t)=32t$.
8. $f(d)=20+0.01d$.
9. $f(w)=.3w+.5$.
10. $f(t)=-2t+8$.
11. $f(t)=5t-11$.

14. What every pupil should be able to do. Having studied Chapter I you should be able to do the following:

1. To translate into algebraic symbols statements expressing direct variation, such as: The distance sound travels varies directly as the time.

2. To name the constants and the variables in equations like $s=\frac{1}{2}gt^2$.

3. To solve simple problems in variation by graph and by proportion.

4. To find a required term of a given arithmetical progression.

5. To find the sum of a given number of terms of an arithmetical progression.

6. To solve simple verbal problems leading to progressions.

7. To find the value of a linear polynomial for given values of the variables and constants.

8. To make the graph of a linear polynomial.

15. Typical problems and exercises. The follow-lowing problems are typical of the work of Chapter I. Every pupil should be able to solve them.

1. In the relation $s=4\pi r^2$ name the constants, the variables.

2. Express in algebraic symbols the statement: The pay a workman receives varies directly as the number of days he works.

3. The weight of a liquid varies as the volume. If 5 cubic feet of water weigh 312 pounds, find the weight of 8 cubic feet.

4. Make a graph of the relation in Exercise 3 and verify your solution by means of the graph.

5. Make a graph of the equation $p = .06b$.

6. A ball rolling down an incline rolls 10 feet the first second. Thereafter in each second it passes over 16 feet more than in the preceding. How far has it rolled in 12 seconds? What is the distance the ball rolls during the 10th second?

CHAPTER II

POLYNOMIALS OF DEGREE HIGHER THAN THE FIRST

Quadratic Polynomials

16. What is meant by a quadratic function. Monomials, binomials, and trinomials like πr^2, $\frac{1}{2}gt^2$, $\sqrt{\frac{3}{4}}a^2$, $s_0+\frac{1}{2}gt^2$, $4x^2+3x+2$, and $3-2x+5x^2$ are *quadratic functions.* A quadratic function contains the variable to the second degree, but no higher.

The function πr^2 may be written $\pi r^2+0\cdot r+0$. This is of the form ax^2+bx+c, where $a=\pi$, $b=0$, $c=0$.

Similarly, $4x^2+3x+2$ is of the form ax^2+bx+c, where $a=4$, $b=3$, $c=2$. Quadratic functions when simplified reduce to the form ax^2+bx+c.

Exercises

1. Show that each of the functions given above is of the form ax^2+bx+c.

2. Make up several quadratic functions and show each to be of the form ax^2+bx+c.

Determine the values of the coefficients a, b, and c for the following functions:

3. x^2+6x+4.

4. $-a^2+3a-5$.

5. $4x-2x^2$.

6. $\frac{1}{2}(3y^2-5y)$.

7. $\frac{1}{4}(x^2+3)$.

8. $-16t^2+50t$.

9. The area of an equilateral triangle is given by the formula $A=\frac{a^2}{4}\sqrt{3}$, where a represents the length of the side. Make the graph.

10. The height s, reached in t seconds by a ball thrown upward vertically with a velocity of 50 feet a second, is found from the equation $s=-16t^2+50t$. Construct the graph.

17. How to make a graph of a quadratic function. If $f(x)$ denotes the quadratic function $2x^2-7x+5$, then $f(2)$ can be found by substituting 2 for x.

$$\begin{aligned}\text{Hence } f(2) &= 2(2)^2-7(2)+5\\ &= 2\cdot 4-14+5\\ &= -1.\end{aligned}$$

Similarly, $f(-2)=2(-2)^2-7(-2)+5=2\cdot 4+14+5$
$=37.$

To make the graph of $f(x)$, find $f(0)$, $f(1)$, $f(2)$, $f(3)$, etc., $f(-1)$, $f(-2)$, $f(-3)$, etc.

Make a table of corresponding values of x and $f(x)$.

Plot the corresponding values of x and $f(x)$ and join the points thus found by a smooth curved line.

Exercise 1, below, illustrates the method:

Exercises

Make the graphs of the following functions and discuss each:

1. $f(x)=2x^2-7x+5$.

Solution: Arrange the corresponding values of x and $f(x)$ in the form of a table.

Plot the number pairs.

Draw a smooth curved line through the points. This gives the graph of $2x^2-7x+5$ (Fig. 4).

x	$f(x)$
0	$2\cdot 0-7\cdot 0+5=5$
1	$2\cdot 1^2-7\cdot 1+5=0$
2	$2\cdot 2^2-7\cdot 2+5=-1$
3	$2\cdot 3^2-7\cdot 3+5=2$
4	$2\cdot 4^2-7\cdot 4+5=9$
.	
.	
.	
-1	$2(-1)^2-7(-1)+5=14$

Discussion: As x increases from 0 to 1, $f(x)$ decreases from 5 to 0. $f(x)$ then becomes negative and continues to decrease until $x=2$. The value of $f(x)$ is then -1.

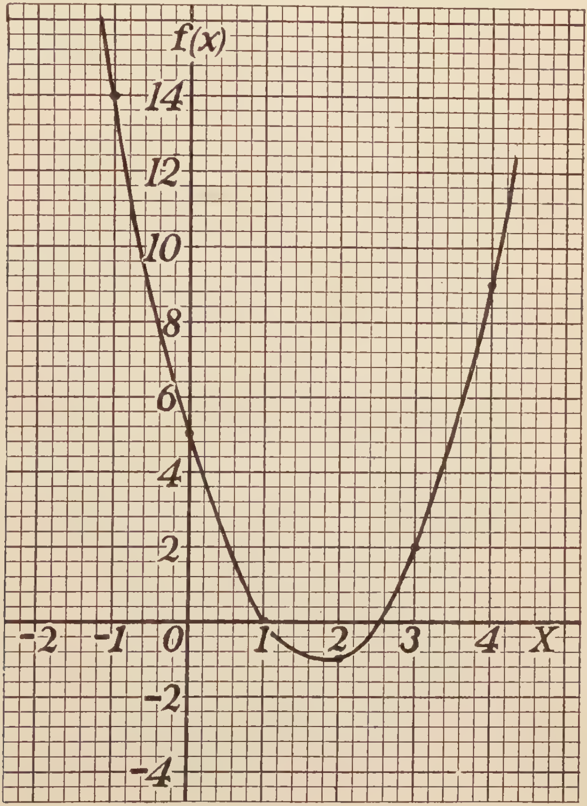


FIG. 4

As x increases from 2 to 3, $f(x)$ increases from -1 to 2.

As x continues to increase without bound, $f(x)$ also increases without bound.

The graph of the quadratic function ax^2+bx+c is called a *parabola*.

2. x^2-6x+5.
3. x^2+2x-8.
4. x^2+4x+3.
5. x^2+3x.
6. $2x^2-x-3$.
7. $2x^2-9x-5$.
8. $-3x^2-2x+16$.
9. $4-5x-x^2$.

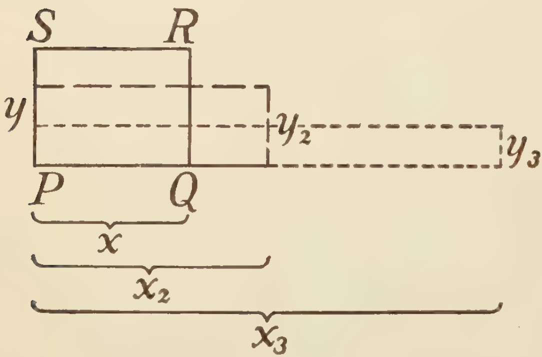


FIG. 5

18. **What is meant by *inverse variation.*** Let the sides x and y of a rectangle $PQRS$ (Fig. 5) vary and let the area A remain constant. Then y must decrease when x increases, and y must increase when x decreases, but the product xy remains the same, always being equal to A. If two variables x and y are so related that their product xy remains constant, either is said to *vary inversely* as the other. Thus, the statements "**x varies inversely as y**" and "**$xy=c$**" are equivalent.

The term *inversely* is used because the equation $xy=c$ may be changed to $x=c\cdot\frac{1}{y}$, which in the language of variation means that x varies as the *inverse* of y. The equation $xy=c$ is a quadratic equation in x and y.

Exercises

Express the following statements in symbols in the form $xy = c$:

1. The time required to do a piece of work varies inversely as the number of men employed.

2. The pressure of a gas on the walls of a retaining vessel varies inversely as the volume.

3. The illumination, on the page of a book, from an incandescent lamp varies inversely as the square of the distance from it.

4. The force of gravitation due to the earth varies inversely as the square of the distance from the earth's center.

19. How to represent the equation $xy = c$ graphically. We have seen (Fig. 5) that as x increases y

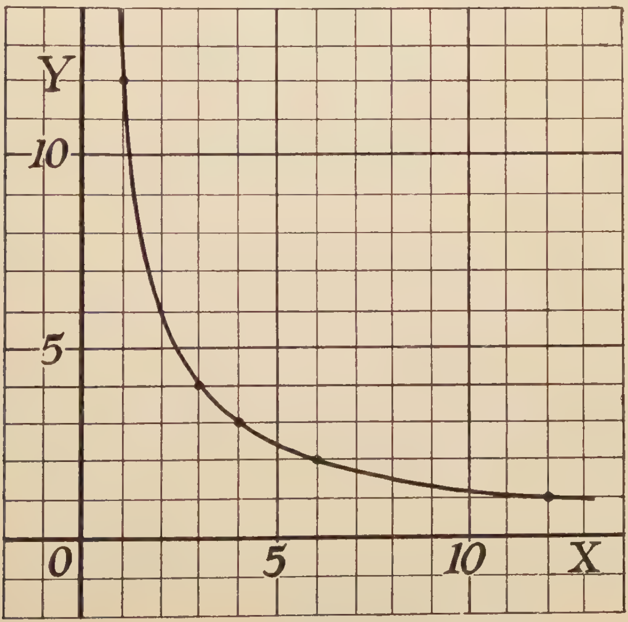


Fig. 6

must decrease in order that the product xy remain constant.

Let $c=12$. Then the equation is $xy=12$.

Let x take the values 1, 2, 3, 4, . . . 12.

Then y takes the corresponding values 12, 6, 4, 3, . . . 1.

Plot the pairs of corresponding values as shown in Fig. 6.

Draw a smooth curve through the marked points.

The curve is the graph of the equation $xy=12$.

Note that the plotted points in Fig. 6 correspond to the upper right-hand corners in the rectangles in Fig. 5.

Exercises

Make graphs for the following equations:

1. $xy=6$.
2. $xy=8$.
3. $xy=24$.
4. $xy=4$.

20. Solving problems in inverse variation by means of proportions. The method is illustrated by the following example:

If 12 men build a fence in 8 days, how long will it take 32 men to do it?

Solution: 1. Denoting the number of men by m and the number of days by d, you have $md=c$ (Exercise 1, §18).

2. Let (m_1, d_1) and (m_2, d_2) be two pairs of values of m and d which satisfy the equation $md=c$.

$$\text{Then } \frac{m_1\,d_1}{m_2\,d_2}=\frac{c}{c}=1.$$

Multiply both members of the equation by m_2d_2.

Then $m_1 d_1 = m_2 d_2$.

By dividing both members of this equation by m_2d_1 you have

$$\frac{m_1 d_1}{m_2 d_1} = \frac{m_2 d_2}{m_2 d_1}.$$

This reduces to $\frac{m_1}{m_2} = \frac{d_2}{d_1}$.

3. Using the last equation you can now solve the problems above by the method shown in §7 (see Exercise 1 below).

Exercises

Solve the following problems by means of proportions:

1. If 12 men build a fence in 8 days, how long does it take 32 men to do it?

Solution: It is given that $m_1 = 12$, $d_1 = 8$, $m_2 = 32$.

Substituting these values in the equation $\frac{m_1}{m_2} = \frac{d_2}{d_1}$ you have

$$\frac{\overset{3}{\cancel{12}}}{\underset{8}{\cancel{32}}} = \frac{d_2}{8}$$

$$\therefore d_2 = 3.$$

2. The rate of traveling a given distance varies inversely as the time. If a train traveling 30 miles an hour passes over the distance between two cities in 4 hours, how long will it take a train making 35 miles an hour?

3. Two boys of unequal weights are making a 12-foot teeterboard balance. They have learned in science that the weight varies inversely as the distance from the turning point to make the board balance. The two boys weigh 125 pounds and 102 pounds respec-

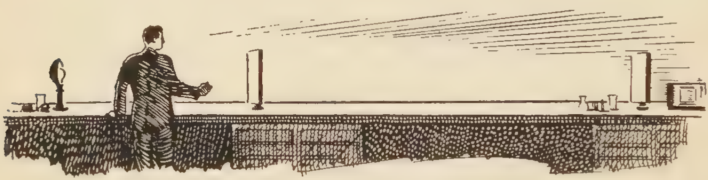
tively. The second boy sits at one end of the board. How far from the other end should the heavy boy sit in order that the board may balance?

4. A lamp shines on a book 5 feet from it. How far from it must the book be held to receive twice as much light? (See Exercise 3, §18.)

5. The attraction of gravitation at points outside the earth's surface varies inversely as the square of the distance from the earth's center. If the attraction on a body at the surface of the earth is 9 pounds, at what height above the surface would the attraction be 4 pounds?

6. If the pressure of a gas on the walls of a retaining vessel is 40 pounds per square foot when the volume is 10 cubic feet, what is the pressure when the volume is 50 cubic feet? (See Exercise 2, §18.)

7. A screen 18 feet from a lamp is moved to a distance of 6 feet from it. How do the intensities compare if the intensity varies inversely as the square of the distance from the screen?



8. The volume of a gas at constant temperature varies inversely as the pressure. If the volume is 150 when the pressure is 30, find the volume when the pressure is 24.

9. An electric current varies inversely as the resistance of the circuit. The current is 3 amperes when the resistance is 5 ohms. Find the current when the resistance is 18.

10. The intensity of light on an object is inversely proportional to the square of the distance of the object from the source of light. If a screen 15 feet from a lamp is moved up a distance of 10 feet, compare the intensities by means of the ratio.

11. The time required to do a piece of work varies inversely as the number of men at work. If 20 men can do a piece of work in 10 days, how long will it take 25 men to do it?

12. The horsepower required to drive a ship varies inversely as the cube of speed. If a horsepower of 2000 propels a ship at the speed of 10 knots, what horsepower is required for a speed of 15 knots?

Polynomials of Degree Higher than the Second

21. A law for finding a polynomial equal to a power of a binomial. The *square* of a binomial, as $1+x$, may be obtained in two ways:

1. You may multiply $1+x$ by $1+x$ by multiplying each term of the first binomial by each term of the second and then adding the similar terms. This gives

$$(1+x)^2=(1+x)(1+x)=1+x+x+x^2=1+2x+x^2.$$

2. You may find the square of $1+x$ briefly by a rule:

Square the first term.
Take twice the product of the two terms.
Square the second term.
Add the three results.

Thus the rule gives directly $(1+x)^2=1+2x+x^2$.

Similarly, $(1-x)^2=1-2x+x^2$.

The *cube* of a binomial may be found by multiplying $1+2x+x^2$ by $1+x$.

This gives:

$$\begin{aligned}(1+x)^3&=(1+2x+x^2)(1+x)\\&=1+2x+x^2\\&+x+2x^2+x^3\\&=1+3x+3x^2+x^3.\end{aligned}$$

The polynomial which is equal to a cube of a binomial may be found directly by the following rule:

Cube the first term.

Take 3 times the square of the first term times the second term.

Take 3 times the first term times the square of the second term.

Cube the second term.

Add the four results.

Exercises

Using the laws above, find the polynomials equal to the following powers:

1. $(1+3a)^2$.

Solution: $(1+3a)^2 = 1+2(1)(3a)+(3a)^2$
$= 1+6a+9a^2$.

2. $(2+y)^3$.

Solution: $(2+y)^3 = 2^3+3(2)^2y+3(2)y^2+y^3$
$= 8+12y+6y^2+y^3$.

3. $(4+a)^2$.
4. $(1+\frac{a}{2})^2$.
5. $(3-x)^2$.
6. $(c+1)^2$.
7. $(p-2)^2$.
8. $(2x+1)^2$.
9. $(3+x^2)^2$.
10. $(a^2-5)^2$.
11. $(1+2x)^3$.
12. $(2-x)^3$.

Suggestion: Since $(2-x)^3$ differs from $(2+x)^3$ only in the $-$ sign before x, the second and fourth terms of the polynomial will be negative.

13. $(2m+1)^3$.
14. $(a-\frac{1}{2})^3$.
15. $(3+x^2)^3$.
16. $(10-2a)^3$.
17. $(a^2-2)^3$.
18. $(1+3b)^3$.
19. $(5+2m)^3$.
20. $(3p-4)^3$.
21. $(4x+3)^3$.

22. Finding values of polynomials. You have seen that the graph of the linear function $ax+b$ is a straight line, and that the graph of the quadratic function ax^2+bx+c is a parabola. Functions of the third degree, as x^3, $2x^3+x^2$, x^3+3x^2-7x+5, are *cubic* functions. The method of making the graph of a cubic function is the same as that employed with linear and quadratic functions.

Thus, to find the value of $f(x)=x^3+3x^2-7x+5$, substitute values for x:

$$\begin{aligned}
\text{Then } f(0) &= 0^3+(3\cdot 0^2)-(7\cdot 0)+5=5\\
f(1) &= 1^3+(3\cdot 1^2)-(7\cdot 1)+5=2\\
f(2) &= 2^3+(3\cdot 2^2)-(7\cdot 2)+5=11\\
f(3) &= 3^3+(3\cdot 3^2)-(7\cdot 3)+5=38\\
f(-3) &= (-3)^3+3(-3)^2-7(-3)+5=26.
\end{aligned}$$

Before making the graph a simpler method of finding the value of $f(x)$ will be shown. It will be seen in §25 how the method is derived.

Let it be required to find $f(2)$:

1. Write down the coefficients of $f(x)$ as shown below. They are 1, 3, -7, 5.

2. Multiply the first coefficient by 2 and add the product to the second. The result is 5.

3. Multiply the result by 2 and add the product to the next coefficient.

4. Multiply the result, 3, by 2 and add the product to the last coefficient. This gives 11, which was found above to be $f(2)$.

This work may be arranged in the following convenient and simple form:

$$\begin{array}{rrrrl|l} 1 & 3 & -7 & 5 & & \underline{\ 2\ } \\ & 2 & 10 & 6 & & \\ \hline 1 & 5 & 3 & \boxed{11} & =f(2) & \end{array}$$

Explain each step in the process below:

$$\begin{array}{rrrrl|l} 1 & 3 & -7 & 5 & & \underline{-2\ } \\ & -2 & -2 & 18 & & \\ \hline 1 & 1 & -9 & \boxed{23} & =f(-2) & \end{array}$$

Using the short process, find $f(3)$, $f(-3)$.

To understand the explanation of the short process of finding the value of a polynomial you must first learn how to divide one polynomial by another (§24).

In finding the value of $f(x)$ by the short process you must be sure that every coefficient is used. For example, in the polynomials x^3+x^2-4, and x^4-1, some of the powers of x are missing. In that case the coefficients of the missing terms are zero, and the polynomials are written

$$x^3+x^2-4=x^3+x^2+(0\cdot x)-4$$
$$\text{and } x^4-1=x^4+(0\cdot x^3)+(0\cdot x^2)+(0\cdot x)-1.$$

Therefore the coefficients of the two polynomials are 1, 1, 0, -4, and 1, 0, 0, 0, -1.

Since the zero coefficients are easily overlooked, you must be very careful not to omit them.

Exercises

Using the short method of §22 find the values of the following functions and check each by the long method of substitution:

1. $f(x)=4x^3+x^2-3x+1$; find $f(-2)$.

2. $f(y)=y^4-3y^3+4y+2$; find $f(3)$.

3. $f(x)=2x^4+6x-8$; find $f(-6)$.

4. $f(y)=5y^3+8y^2-4$; find $f(2)$.

5. $f(a)=a^4+4a^2+16$; find $f(-1)$.

6. $f(t)=21t^2-23t-6$; find $f(-3)$.

7. $f(x)=x^3-7x-3$; find $f(-2)$.

23. Making the graphs of polynomials. In making the table of corresponding numbers the short method of finding values of polynomials (§22) is very helpful. The first exercise below illustrates how to make the graph.

EXERCISES

Make a graph of each of the following polynomials:

1. $2x^3-5x^2-14x+8$.

Find the values of the polynomial that correspond to given values of x:

Solution:

2	−5	−14	+8	2
	4	−2	−32	
2	−1	−16	−24	
2	−5	−14	+8	3
	6	3	−33	
2	−1	−11	−25	
2	−5	−14	+8	4
	8	+12	−8	
2	3	−2	0	
2	−5	−14	+8	5
	10	25	55	
2	5	11	63	
2	−5	−14	+8	−1
	−2	7	7	
2	−7	−7	15	

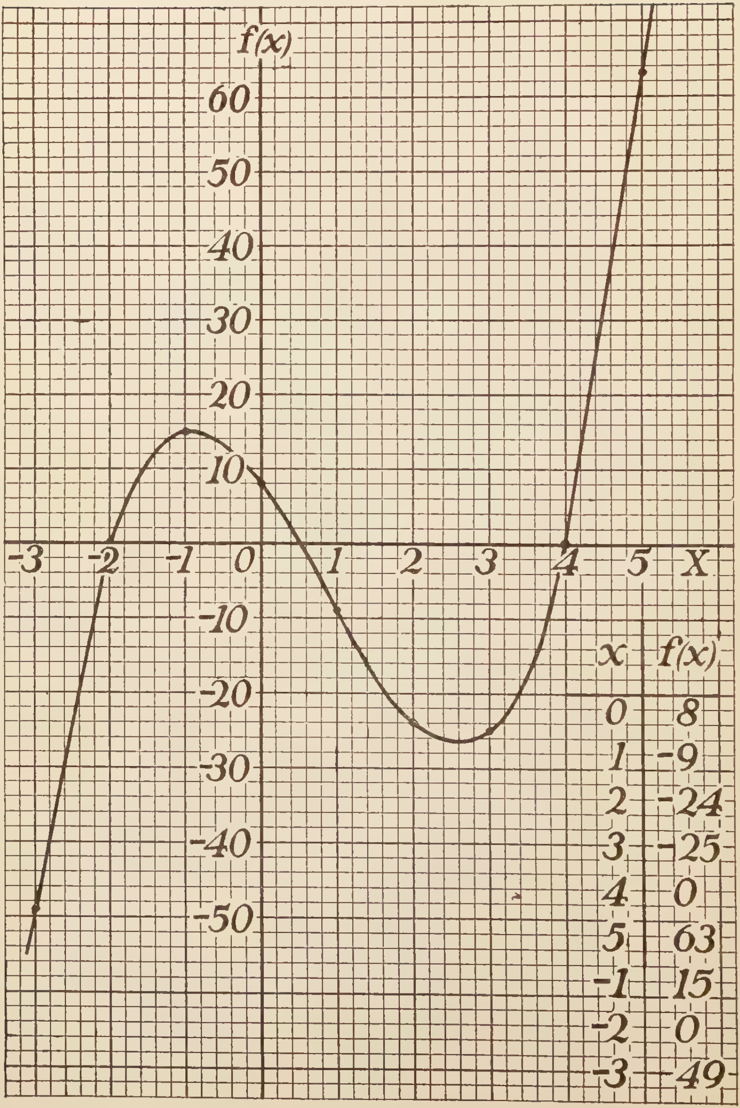

f(x)
60
50
40
30
20
10
-10
-20
-30
-40
-50
-3 -2 -1 0 1 2 3 4 5 X
x f(x)
0 8
1 -9
2 -24
3 -25
4 0
5 63
-1 15
-2 0
-3 -49

Fig. 7

2	-5	-14	$+8$	$\lfloor -2$
	-4	18	-8	
2	-9	4	$\boxed{0}$	

2	-5	-14	$+8$	$\lfloor -3$
	-6	$+33$	-57	
2	-11	19	$\boxed{-49}$	

Tabulate the corresponding number pairs as shown in Fig. 7.
Plot the points corresponding to the number pairs.
Draw the graph.

2. x^3-5x^2+2x+8.

3. x^3-6x^2+7x+4.

4. x^3-3x^2-3x+1.

24. Dividing one polynomial by another. The process of dividing polynomials is the same as that of dividing arithmetical numbers. In fact, an arithmetical number may be considered a special case of an algebraic polynomial.

Thus, when $x=10$, the polynomial $6x^3+7x^2+5x+2$ $=6\cdot 10^3+7\cdot 10^2+5\cdot 10+2=6000+700+50+2=6752$.

To illustrate the process of dividing polynomials the division of 6752 by 32 will be arranged in several ways:

(*a*)

6752	32
64	211
35	
32	
32	
32	

(*b*)

$6000+700+50+2$	$30+2$
$6000+400$	$200+10+1$
$300+50$	
$300+20$	
$30+2$	
$30+2$	

(*c*)

$$
\begin{array}{llll}
6\cdot10^3+7\cdot10^2+5\cdot10+2 & \multicolumn{1}{|l}{3\cdot10+2} \\ \cline{2-2}
\underline{6\cdot10^3+4\cdot10^2} & \multicolumn{1}{|l}{2\cdot10^2+1\cdot10+1} \\
\quad\quad 3\cdot10^2+5\cdot10 & \\
\quad\quad \underline{3\cdot10^2+2\cdot10} & \\
\quad\quad\quad\quad 3\cdot10+2 & \\
\quad\quad\quad\quad \underline{3\cdot10+2} &
\end{array}
$$

(*d*)

$$
\begin{array}{ll}
6x^3+7x^2+5x+2 & \multicolumn{1}{|l}{3x+2} \\ \cline{2-2}
\underline{6x^3+4x^2} & \multicolumn{1}{|l}{2x^2+x+1} \\
\quad\quad 3x^2+5x & \\
\quad\quad \underline{3x^2+2x} & \\
\quad\quad\quad\quad 3x+2 & \\
\quad\quad\quad\quad \underline{3x+2} &
\end{array}
$$

The last example (*d*) shows that in dividing one polynomial by another the following steps are taken:

1. Divide the first term of the divisor into the first term of the dividend. This gives the first term of the quotient . . . . $6x^3+7x^2+5x+2 \,\big|\, 3x+2$; $2x^2+x+1$

2. Multiply the divisor by the first term of the quotient. . . . . $\underline{6x^3+4x^2}$

3. Subtract and bring down the next term . . . . . . $3x^2+5x$

4. Divide the first term of the divisor into the first term of the remainder.

This gives the second term of the quotient.

5. Multiply the divisor by the second term of the quotient . . . . $\underline{3x^2+2x}$

6. Subtract and bring down the next term . . . . . . $3x+2$

7. Repeat steps 4, 5, and 6 . . . . $\underline{3x+2}$

Exercises

Divide as indicated:

1. $(x^3+2x^2-x-2)\div(x+1)$.
2. $(x^3-18x-35)\div(x-5)$.
3. $(x^6+4x^4+x^2-6)\div(x^2+2)$.
4. $(x^6-6x^4-19x^2+84)\div(x^2+4)$.
5. $(6x^4+19x^3+16x^2+3x-12)\div(3x^2+5x-4)$.
6. $(3x^5+3x^4-11x^3+x^2+4x)\div(3x^2+9x+4)$.
7. $(x^3+6x^2+5x-12)\div(x+3)$.
8. $(2x^5+8x^4-x^3-26x^2-13x+6)\div(x^2+4x+3)$.
9. $(x^4-1)\div(x-1)$.
10. $(x^5-1)\div(x-1)$.
11. $(6a^3-a^2b-2ab^2-15b^3)\div(2a-3b)$.
12. $(12m^3a^3-17m^2a^2+10ma-3)\div(4ma-3)$.
13. $(12m^4-25m^3n+12m^2n^2)\div(3m^2-4mn)$.
14. $(a^2-b^2-2bc-c^2)\div(a-b-c)$.
15. $(a^5-b^5)\div(a-b)$.
16. $(a^3-3a^2b+3ab^2-b^3)\div(a-b)$.

25. The meaning of synthetic division.* Synthetic division is an abbreviated form of long division. Like long division, it is used to find the quotient and the remainder. In synthetic division only the coefficients are used and only the coefficients of the quotient are determined. The process was used in §22 without an explanation of the method by which it is derived. This will be given below.

When the polynomial $6x^3-10x^2-50x-21$ is divided by $x-4$, the degree of the quotient is one lower than that of the dividend. The quotient is therefore of the quadratic form ax^2+bx+c. As soon as the coefficients a, b, and c are known, the quotient ax^2+bx+c can be stated.

In the six steps below it is shown how to find the coefficients a, b, and c first by long division, and then by the short process of synthetic division.

a. The following is the complete process of long division:

$$
\begin{array}{rl}
6x^3-10x^2-50x-21 & \underline{|\, x-4} \\
\underline{6x^3-24x^2} & \underline{|\, 6x^2+14x+6} \\
14x^2-50x & \\
\underline{14x^2-56x} & \\
6x-21 & \\
\underline{6x-24} & \\
3 &
\end{array}
$$

b. If the literal number x is omitted, the process reduces to the following:

*Future work does not depend on §25. Therefore it may be omitted, or assigned for study to individual pupils.

$$
\begin{array}{rrrrl}
6 & -10 & -50 & -21 & \underline{|\,-4\quad} \\
\underline{6} & \underline{-24} & & & \underline{|\,6+14+6} \\
 & 14 & -50 & & \\
 & \underline{14} & \underline{-56} & & \\
 & & 6 & -21 & \\
 & & \underline{6} & \underline{-24} & \\
 & & & 3 &
\end{array}
$$

c. If the terms of the dividend are not brought down in subtracting the partial products, the process is further reduced:

$$
\begin{array}{rrrrl}
6 & -10 & -50 & -21 & \underline{|\,-4\quad} \\
\underline{6} & \underline{-24} & & & \underline{|\,6+14+6} \\
 & 14 & & & \\
 & \underline{14} & \underline{-56} & & \\
 & & 6 & & \\
 & & \underline{6} & \underline{-24} & \\
 & & & 3 &
\end{array}
$$

d. Since subtraction means to change the sign of the subtrahend and to add, the divisor $x-4$ is changed to $x+4$, or correspondingly -4 is changed to 4. This changes the sign of every second term in the partial product, which is then added to the number above:

$$
\begin{array}{rrrrl}
6 & -10 & -50 & -21 & \underline{|\,4\quad} \\
\underline{6} & \underline{24} & & & \underline{|\,6+14+6} \\
 & 14 & & & \\
 & \underline{14} & \underline{56} & & \\
 & & 6 & & \\
 & & \underline{6} & \underline{24} & \\
 & & & 3 &
\end{array}
$$

e. The first terms in the partial products are now omitted since they are exactly the same as the numbers above:

$$
\begin{array}{rrrr|l}
6 & -10 & -50 & -21 & \underline{4} \\
 & 24 & & & \underline{6+14+6} \\
\hline
 & 14 & & & \\
 & & 56 & & \\
\cline{2-4}
 & & 6 & & \\
 & & & 24 & \\
\cline{3-4}
 & & & 3 &
\end{array}
$$

f. Finally, by moving all the numbers the process is condensed to the simple form:

$$
\begin{array}{rrrr|l}
6 & -10 & -50 & -21 & \underline{4} \\
 & 24 & 56 & 24 & \\
\hline
6 & 14 & 6 & \boxed{3} &
\end{array}
$$

Note that the first three terms in the lowest line are the coefficients of the quotient. Hence the quotient has been omitted.

From form (*f*) it is possible to derive a rule for dividing a polynomial $f(x)$ by $x-a$, where a is any number:

1. *Arrange* $\boldsymbol{f(x)}$ *according to descending powers of* $\boldsymbol{x}$, *supplying zero coefficients for the missing terms.*

2. *Bring down the coefficient of the first term of the dividend, multiply it by* $\boldsymbol{a}$, *and add the product to the second term of the dividend.*

3. *Repeat the process,* i.e., *multiply the last sum by* $\boldsymbol{a}$ *and add the product to the third term of the dividend.*

4. *Repeat again, until a product has been added to the last term.*

5. Note that the first sums are the coefficients of the quotient and that the last sum is the remainder. It will be seen in the Exercises below that the last sum is also the value of the polynomial when $x=4$.

This process of dividing $f(x)$ by $x-a$ is called *synthetic division.*

Exercises

Divide as indicated:

1. $(x^4-2x^3+3x-2)\div(x-3)$.

a. Method 1 (long division):

$$
\begin{array}{l|l}
x^4-2x^3+3x-2 & x-3 \\
\hline
\underline{x^4-3x^3} & x^3+x^2+3x+12 \\
\quad x^3 & \\
\quad \underline{x^3-3x^2} & \\
\qquad 3x^2+3x & \\
\qquad \underline{3x^2-9x} & \\
\qquad\quad 12x-\ \ 2 & \\
\qquad\quad \underline{12x-36} & \\
\qquad\qquad\quad 34=\text{remainder.} &
\end{array}
$$

Method 2 (synthetic division):

$$
\begin{array}{cccccl}
1 & -2 & 0 & 3 & -2 & \lfloor 3 \\
 & 3 & 3 & 9 & 36 & \\
\hline
1 & 1 & 3 & 12 & \boxed{34} & =\text{remainder.}
\end{array}
$$

$$\therefore \frac{x^4-2x^3+3x-2}{x-3}=(x^3+x^2+3x+12)+\frac{34}{x-3}.$$

b. If $f(x)=x^4-2x^3+3x-2$, find $f(3)$ by substituting 3 for x. Compare $f(3)$ with the remainder found above.

2. Divide $3x^3-6x+3$ by $x-1$.
Find $f(1)$ by substituting 1 for x and compare it with the remainder.

3. Divide $4x^3+2x^2-7$ by $x-5$.
Find $f(5)$ and compare it with the remainder.

4. Divide x^5-1 by $x-1$.
Find $f(1)$ and compare it with the remainder.

5. Divide $2x^2-3x+5$ by $x-a$, a being any number.
Compare $f(a)$ with the remainder.

6. Divide $3x^3-x^2+2x+5$ by $x+2$.

Suggestion: If synthetic division is used, $x+2$ must be changed to $x-(-2)$. The function is then divided synthetically by -2.
Compare $f(-2)$ with the remainder.

7. Divide $5x^4+3x-5$ by $x+3$.
Compare $f(-3)$ with the remainder.

26. The remainder theorem.* Exercises 1 to 7 above illustrate the following theorem: *If a polynomial in $\boldsymbol{x}$ is divided by $\boldsymbol{x-a}$, where $\boldsymbol{a}$ is any number, the remainder is the same as the value of the polynomial for $\boldsymbol{x=a}$.* Briefly we may say, *if $\boldsymbol{f(x)}$ is divided by $\boldsymbol{x-a}$, the remainder is $\boldsymbol{f(a)}$.* This explains the use of synthetic division in finding the value of polynomials.

Exercises

Find the value of each of the following polynomials for the value of x indicated:

1. $x^4-2x^3+3x^2-x+7$ for $x=3$.
2. $3x^3-4x^2-3x+5$ for $x=2$.
3. x^3-2x^2-7x-4 for $x=-2$.
4. $3x^3-4x+7$ for $x=-3$.

Suggestion: Denote the coefficient of x^2 by 0.

5. $x^3+6x^2+11x+6$ for $x=-3$.

*To be omitted if §25 has not been studied.

6. $3x^3-14x+1$ for $x=2$.

7. $x^3+2x^2-9x+18$ for $x=-3$.

How to Solve Equations Graphically

27. Locating points at which the graph of a polynomial crosses the x-axis. The polynomial $f(x)=x^3-5x^2+2x+8$ has the value zero for $x=2$. Graphically this means that the graph of x^3-5x^2+2x+8 crosses the x-axis at $x=2$. Algebraically it is said that 2 is a *solution,* or *root,* of the equation $x^3-5x^2+2x+8=0$. The graph enables you to solve the equation as follows:

1. Make the graph of $f(x)$.

2. Locate the points of intersection of the graph and the x-axis.

3. Measure the distances from the origin to the points of intersection.

4. These lengths are the roots of the equation $f(x)=0$.

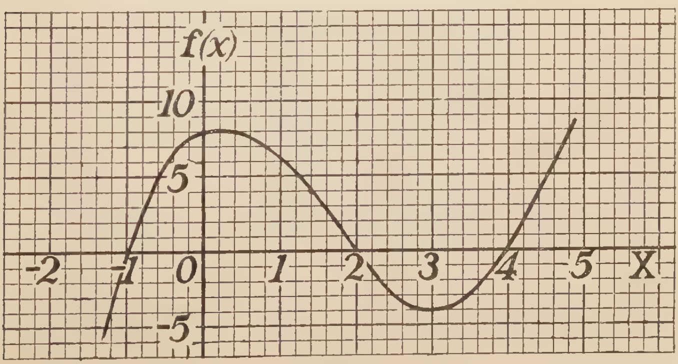


Fig. 8

Exercises

1. Show by use of Fig. 8 that the roots of the equation $x^3-5x^2+2x+8=0$ are $x_1=2$, $x_2=4$, $x_3=-1$.

Solve the equations in Exercises 2 to 8 graphically:

2. $x^2-6x+8=0$.
3. $x^3-6x^2+11x-6=0$.
4. $x^3-2x^2-7x-4=0$.
5. $x^3-13x-12=0$.
6. $x^3-7x^2+15x-9=0$.
7. $x^3-3x^2-10x+24=0$.
8. $12x^3-11x^2-13x+10=0$.
9. $x^3-9x^2+23x-15=0$.
10. $x^3-2x^2-18x+24=0$.

28. What every pupil should be able to do. The chapter has taught you to do the following:

1. To recognize a quadratic polynomial.
2. To make graphs of polynomials of the second degree or higher.
3. To translate into symbols statements of inverse variation.
4. To solve problems in inverse variation.
5. To express a second or third power of a binomial as a polynomial.
6. To divide one polynomial by another.
7. To solve graphically equations of the second degree or higher.

29. Typical problems and exercises. Solve the following:

1. Make a graph of x^2+3x and from the graph find the roots of the equation $x^2+3x=0$.

2. Make a graph of $4-5x-x^2$.

3. Divide $6x^4+19x^3+13x^2-2x-8$ by $2x^2+3x+2$.

4. Solve the equation $2x^3-3x^2-14x+15=0$.

5. State in algebraic symbols that x varies inversely as z.

6. When air is pumped into a container, the pressure on the surface varies inversely as the volume. When the volume is 25 cubic feet the pressure is 80 pounds. Find the pressure when the volume is 3 cubic feet.

7. State without multiplying, the polynomial equal to $(2x-3)^3$.

8. Prepare a floor talk on one of the following topics.

a. The laws of variation.

b. Representing algebraic polynomials graphically.

c. The use of graphs in solving equations.

CHAPTER III

TRIGONOMETRIC RATIOS

Relations between the Sides and Angles of a Right Triangle

30. What is meant by the sine ratio. In determining unknown distances by means of the right triangle you have made use of the tangent ratio. A table was worked out which gives the values of tangent ratios corresponding to given angles. Relationships between the sides and acute angles of a right triangle are useful in surveying and in the sciences. In this chapter a further study is to be made of these relations.

Exercises

1. On squared paper (Fig. 9) draw angle ABC equal to 23°.

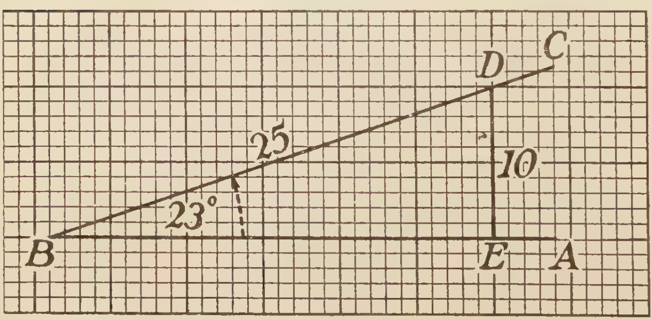


Fig. 9

From a point D on BC draw $DE \perp BA$.

By measuring the side opposite angle B show that $DE = 10$.

Similarly, measure the hypotenuse BD. This gives $BD = 25$.

By dividing show that the ratio $\frac{DE}{BD} = \frac{10}{25} = .40$.

Compare your result with those of other members of the class.

The ratio obtained by dividing the side opposite B in the triangle ABC by the hypotenuse is called the *sine ratio*. The word "sine" is usually abbreviated "sin," and "sine of 23°" is written briefly "sin 23°."

2. Explain by means of a principle of similar triangles why all pupils of the class should find the same value for sin 23°.

3. On squared paper draw an angle equal to 30°. As in Exercise 1 draw a right triangle containing the 30° angle. Measure the side opposite the 30° angle, measure the hypotenuse, and find the ratio of the first to the second. What is the sine of 30°? Compare your result with those of other pupils.

4. As in Exercises 1 and 2, find sin 67°.

31. A table of sine ratios. If you draw many angles of various sizes and find the sine of each by the method explained in the exercises above, you may make a table of corresponding values of angles and sine ratios. Such a table would be called a *table of sines*. The table on the following page contains the angles in the first column and their sines in the second. From the table find the sines of 23°; 30°; 67°.

The values in the table have been computed to 4 places of decimals. Your results should agree with them to 2 decimal places. You cannot draw and measure accurately enough to determine the correct third figure.

Table of Sines, Cosines, and Tangents of Angles from 1° to 90°

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2	.0349	.9994	.0349	47	.7314	.6820	1.0724
3	.0523	.9986	.0524	48	.7431	.6691	1.1106
4	.0698	.9976	.0699	49	.7547	.6561	1.1504
5	**.0872**	**.9962**	**.0875**	**50**	**.7660**	**.6428**	**1.1918**
6	.1045	.9945	.1051	51	.7771	.6293	1.2349
7	.1219	.9925	.1228	52	.7880	.6157	1.2799
8	.1392	.9903	.1405	53	.7986	.6018	1.3270
9	.1564	.9877	.1584	54	.8090	.5878	1.3764
10	**.1736**	**.9848**	**.1763**	**55**	**.8192**	**.5736**	**1.4281**
11	.1908	.9816	.1944	56	.8290	.5592	1.4826
12	.2079	.9781	.2126	57	.8387	.5446	1.5399
13	.2250	.9744	.2309	58	.8480	.5299	1.6003
14	.2419	.9703	.2493	59	.8572	.5150	1.6643
15	**.2588**	**.9659**	**.2679**	**60**	**.8660**	**.5000**	**1.7321**
16	.2756	.9613	.2867	61	.8746	.4848	1.8040
17	.2924	.9563	.3057	62	.8829	.4695	1.8807
18	.3090	.9511	.3249	63	.8910	.4540	1.9626
19	.3256	.9455	.3443	64	.8988	.4384	2.0503
20	**.3420**	**.9397**	**.3640**	**65**	**.9063**	**.4226**	**2.1445**
21	.3584	.9336	.3839	66	.9135	.4067	2.2460
22	.3746	.9272	.4040	67	.9205	.3907	2.3559
23	.3907	.9205	.4245	68	.9272	.3746	2.4751
24	.4067	.9135	.4452	69	.9336	.3584	2.6051
25	**.4226**	**.9063**	**.4663**	**70**	**.9397**	**.3420**	**2.7475**
26	.4384	.8988	.4877	71	.9455	.3256	2.9042
27	.4540	.8910	.5095	72	.9511	.3090	3.0777
28	.4695	.8829	.5317	73	.9563	.2924	3.2709
29	.4848	.8746	.5543	74	.9613	.2756	3.4874
30	**.5000**	**.8660**	**.5774**	**75**	**.9659**	**.2588**	**3.7321**
31	.5150	.8572	.6009	76	.9703	.2419	4.0108
32	.5299	.8480	.6249	77	.9744	.2250	4.3315
33	.5446	.8387	.6494	78	.9781	.2079	4.7046
34	.5592	.8290	.6745	79	.9816	.1908	5.1446
35	**.5736**	**.8192**	**.7002**	**80**	**.9848**	**.1736**	**5.6713**
36	.5878	.8090	.7265	81	.9877	.1564	6.3138
37	.6018	.7986	.7536	82	.9903	.1392	7.1154
38	.6157	.7880	.7813	83	.9925	.1219	8.1443
39	.6293	.7771	.8098	84	.9945	.1045	9.5144
40	**.6428**	**.7660**	**.8391**	**85**	**.9962**	**.0872**	**11.4301**
41	.6561	.7547	.8693	86	.9976	.0698	14.3006
42	.6691	.7431	.9004	87	.9986	.0523	19.0811
43	.6820	.7314	.9325	88	.9994	.0349	28.6363
44	.6947	.7193	.9657	89	.9998	.0175	57.2900
45	**.7071**	**.7071**	1.0000	90	1.0000	.0000	∞

Exercises

1. Using the table find the sines of 10°; 25°; 45°; 80°.

2. Using the table find the angles whose sines are .8290; .7314; .4848; .2250.

32. Three important trigonometric ratios. If A (Fig. 10) represents any angle, and if from any point B, on either side of the angle, a perpendicular BC is drawn to the other side, a right triangle ABC is formed.

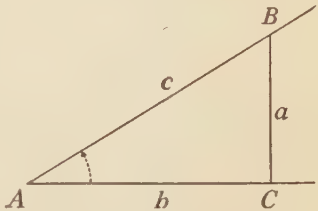


Fig. 10

The following three ratios are important:

1. The ratio of the side opposite point A to the hypotenuse. It is called the *sine of A*.

2. The ratio of the side passing through A to the hypotenuse. It is called the *cosine of A*.

3. The ratio of the side opposite A to the side passing through A. It is called the *tangent of A*.

Briefly, 1. $\sin A = \dfrac{\text{side opposite } A}{\text{hypotenuse}} = \dfrac{a}{c}$.

2. $\cos A = \dfrac{\text{side passing through } A}{\text{hypotenuse}} = \dfrac{b}{c}$.

3. $\tan A = \dfrac{\text{side opposite } A}{\text{side passing through } A} = \dfrac{a}{b}$.

33. How to find by measurement the value of the cosine and tangent of an angle. The method of finding

the tangent or cosine of an angle is the same as that used in finding the sine in Exercises 1 and 3 (§30).

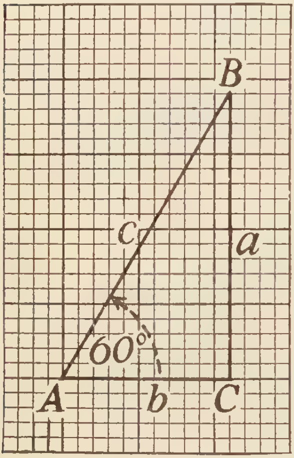


FIG. 11

Thus, to find sin 60°, cos 60°, and tan 60°, proceed as follows:

On squared paper (Fig. 11) draw $A=60°$.

Draw the $BC \perp AC$.

Determine a, b, c, by measurement.

Find $\sin 60° = \frac{a}{c} = \frac{19}{22} = .86$

$\cos 60° = \frac{b}{c} = \frac{11}{22} = .50$

$\tan 60° = \frac{a}{b} = \frac{19}{11} = 1.72.$

Check your results by means of the table (§31).

EXERCISES

Find by measurement the values of the following and check the results by means of the table:

1. sin 40°; cos 40°; tan 40°.
2. sin 70°; cos 70°; tan 70°.
3. sin 64°; cos 64°; tan 64°.

THE USE OF TRIGONOMETRIC RATIOS IN PROBLEMS

34. Problems in finding distances and angles. The trigonometric table can be used to simplify geometric problems. The following exercises illustrate the method used by the surveyor:

Exercises

1. Find the height of the flagstaff of your school.

Solution: To find the height of the flagstaff AB (Fig. 12) some of the ninth grade pupils marked off a distance $CD = 20.5$ feet.

They placed a transit directly over D and measured the angle FEA. They found

$$\angle FEA = 74°$$
$$CB = 2.5 \text{ feet}$$
$$\text{and } FB = 3 \text{ feet}$$

The class used these measurements to determine AB as follows:

Denoting AF by h the pupils wrote

$$\tan 74° = \frac{h}{23}.$$

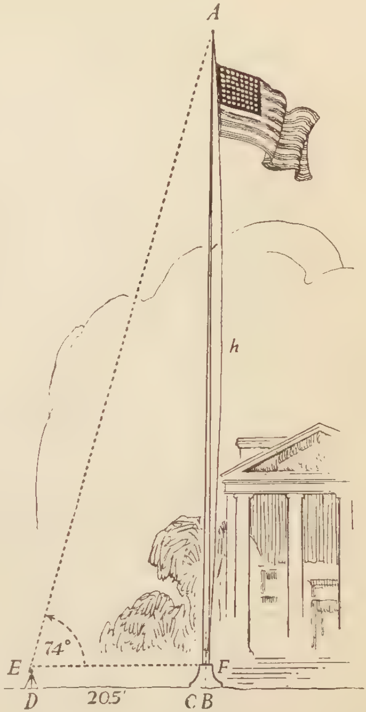


Fig. 12

From the table they found

$$\tan 74° = 3.4874$$

$$\therefore \frac{h}{23} = 3.4874$$

$$\therefore h = (3.49)\,(23) = 80.27$$

$$\therefore AB = 3 + 80.27 = 83.27.$$

2. A telephone pole is braced by a wire fastened to the ground at a point 16 feet from the foot of the pole. The wire makes an angle of 62° with the ground. How high is the pole?

3. A surveyor wishes to measure the distance AB across a pond (Fig. 13). With the help of his transit he lays off $AC \perp AB$ making it 300 feet long. He then measures angle ACB and finds it to be 64°. How long is AB?

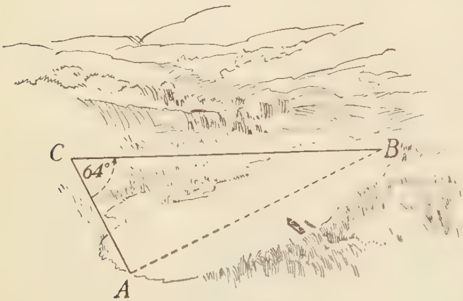


FIG. 13

4. The angle of elevation of the top of a mountain (Fig. 14) is 38°. The distance from the point of observation to the top is 4865 feet. Determine the height of the mountain.

5. A balloon B (Fig. 15) is anchored to the ground at a point P. The point D is directly below B.

FIG. 14

If $\angle DPB = 54°$, $PD = 310$ feet, what is the length of the wire PB? How high is the balloon?

FIG. 15

6. A concrete road is inclined to the horizontal at an

angle of 60°. How much does the road rise for a distance of 100 feet measured along the road?

7. A monument 380 feet high casts a shadow 215 feet long. Find the angle of elevation of the sun.

8. A balloon is fastened by a cable 640 feet long. The angle of elevation of the balloon is 68°. How high is the balloon?

9. A road rises 3.5 feet for every 100 feet measured along the road. What is the angle of elevation of the road?

10. From the top of a cliff 165 feet high the angle of depression of a point marked by a rock is 30°. How far is the rock from the foot of the cliff?

11. The angle through which a pendulum swings is 9 degrees. If the pendulum is 39.1 inches long what is the distance between the two extreme positions?

12. A ship sails N.E. by E. at a rate of 10 knots per hour. At what rate is it moving northward?

13. The shadow of a vertical pole 32 feet high is 50 feet long. What is the sun's altitude?

14. The side of an inscribed regular pentagon is 3.4 feet. Find the radius of the circumscribed circle.

15. The angle of elevation of the highest point of a building is 30° when observed from a point 900 feet from the base of the building. How high is the building?

35. How to find the values of the ratios when the angles are expressed in degrees and minutes. The table (§31) gives the values of the trigonometric ratios only for angles expressed in degrees. For example, it does not give the sine of an angle equal to 30° 20′. It shows that sin 30° 20′ lies between .5000 and .5150. For practical purposes it is assumed that a small change in the sine is nearly proportional to the

change of the angle. Then it may be said that sin 30° 20′ is equal to sin 30° plus $\frac{20}{60}$ of the difference between sin 30° and sin 31°.

Denote by x the number which added to the fractional part of sin 30° gives sin 30° 20′.

$$\text{Then } \frac{20}{60}=\frac{x}{.0150}$$

$$\therefore x=\tfrac{1}{3}(.0150)=.0050$$

$$\therefore \sin 30^\circ\ 20'=.5000+.0050,$$

$$\text{or } \sin 30^\circ\ 20'=.5050.$$

The process which was used to find sin 30° 20′ from sin 30° and sin 31° is called *interpolation.*

Exercises

1. Find tan 68° 40′.

Solution:

$$\tan 68^\circ=2.4751$$
$$\tan 69^\circ=2.6051$$
$$\text{Difference}=\ .1300$$
$$\therefore \frac{40}{60}=\frac{x}{.1300}$$
$$\therefore x=\tfrac{2}{3}(.1300)=.0866$$
$$\therefore \tan 68^\circ\ 40'=2.4751+.0866$$
$$=2.5617.$$

2. Find cos 42° 16′.

Solution:

$$\cos 42^\circ=.7431$$
$$\cos 43^\circ=.7314$$
$$\text{Difference}=.0117$$
$$\therefore \frac{16}{60}=\frac{x}{.0117}$$
$$\therefore x=\frac{16(.0117)}{60}=.0031.$$

Since cos 43° is *less* than cos 42° we must *subtract* x.
Hence, $\cos 42°\ 16' = .7431 - .0031$
$= .7400.$

Find the following values:

3. sin 18° 10′.	**6.** cos 10° 32′.	**9.** tan 15° 21′.
4. cos 21° 40′.	**7.** sin 74° 54′.	**10.** cos 71° 45′.
5. tan 34° 50′.	**8.** tan 28° 18′.	**11.** sin 89° 12′.

12. From a point 210 feet from the foot of a wireless telegraph mast the angle of elevation of the top was 48° 14′. Find the height of the mast.

13. The top of a mountain 12,340 feet high is observed to be at an angle of elevation of 30° 20′ from a station located at an altitude of 5364 feet. What is the direct distance from the station to the mountain top?

14. From the top of a tower 220 feet high the angle of depression of one object in the plane below is found to be 50° 12′. Find the distance from the object to the foot of the tower.

15. To find the width of a stream a surveyor lays off along the bank a line AB 400 feet long. At B he determines a point C on the opposite bank so that $CB \perp AB$. Angle BAC is then measured and found to be 53° 18′. How wide is the stream?

16. A steamer is driven northward by the wind with a force sufficiently great to carry it 10 miles in one hour. The engine is driving it eastward with a force of 14 miles per hour. What distance will it actually travel in one hour and in what direction?

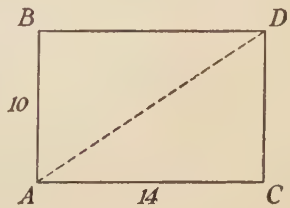


FIG. 16

Solution: Let AB and AC (Fig. 16) represent the magnitude and direction of the two forces. Then it is shown by experiment that the diagonal AD (resultant) of the rectangle $ACDB$ represents the actual rate and direction of the boat. Hence, you must find AD and $\angle DAB$.

17. Two forces AB and AC act at right angles to each other. Find the resultant force AD if AB represents a force of 12 pounds and AC one of 38 pounds. Find the angle which the resultant makes with the second force.

18. A ship sails N.W. by W. at the rate of 8.5 knots per hour. Find at what rate it is moving north.

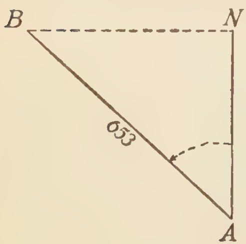


FIG. 17

19. A ship sails from latitude 48° 20′ N. along a course AB (Fig. 17) running N.W. a distance of 653 nautical miles. Find the latitude arrived at.

Suggestion: One nautical mile is approximately 1 minute. Find NA in miles, change it to degrees and minutes, and add that to 48° 20′.

RELATIONS BETWEEN TRIGONOMETRIC RATIOS*

36. A relation between the sine and cosine of an angle. When the sine of an angle is known the cosine is found from a relation worked out below:

Let $\sin A = \frac{a}{c}$ (Fig. 18).

Then $\cos A = \frac{b}{c}$.

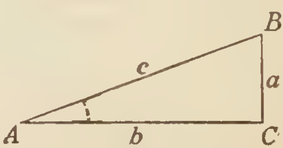


FIG. 18

Squaring each equation, you have

$$(\sin A)^2 = \frac{a^2}{c^2}$$

$$(\cos A)^2 = \frac{b^2}{c^2}$$

$$\therefore (\sin A)^2 + (\cos A)^2 = \frac{a^2+b^2}{c^2}.$$

*§§36 and 37 may be omitted or assigned as supplementary topics to special pupils.

Since $a^2+b^2=c^2$,
show that
$(\sin A)^2+(\cos A)^2=1.$

The last equation is usually written

$$\sin^2 A+\cos^2 A=1.$$

EXERCISES

1. Using the equation $\sin^2 x+\cos^2 x=1$, find $\cos x$ if $\sin x=\frac{1}{2}$.

Solution:
$$\sin^2 x+\cos^2 x=1$$
$$\therefore \tfrac{1}{4}+\cos^2 x=1$$
$$\cos^2 x=\tfrac{3}{4}$$
$$\cos x=\frac{\sqrt{3}}{2}.$$

2. Find the value of x in Exercise 1.

3. Find $\sin A$, if $\cos A=\frac{1}{4}$.

4. Find $\cos y$, if $\sin y=\frac{1}{2}\sqrt{3}$.

5. Find $\sin B$, if $\cos B=\frac{1}{3}$.

6. Find $\cos B$, if $\sin B=\frac{1}{4}$.

7. Find $\sin B$, if $\cos B=.5$.

37. A relation between the sine, cosine, and tangent of an angle. Let A be one of the acute angles of the right triangle ABC (Fig. 19).

Then $\sin A=\dfrac{a}{c}$

$$\cos A=\frac{b}{c}$$

$$\therefore \frac{\sin A}{\cos A}=\frac{\frac{a}{c}}{\frac{b}{c}}=\frac{a}{b}.$$

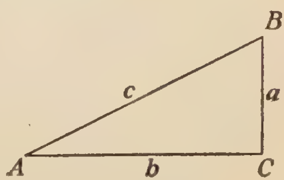


FIG. 19

Since $\tan A = \dfrac{a}{b}$, it follows that

$$\tan A = \frac{\sin A}{\cos A}.$$

EXERCISES

1. Cos $A = \frac{1}{2}$. Find tan A.

Solution:
$$\sin^2 A + \cos^2 A = 1$$
$$\therefore \sin^2 A + \tfrac{1}{4} = 1$$
$$\sin^2 A = \tfrac{3}{4}$$
$$\sin A = \frac{\sqrt{3}}{2}$$
$$\therefore \tan A = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}.$$

2. Sin $A = \frac{1}{3}$. Find tan A.

3. Cos $A = \frac{3}{4}$. Find tan A.

4. Sin $A = \frac{1}{2}\sqrt{3}$. Find tan A.

38. What every pupil should be able to do. Every pupil who has studied Chapter III should be able to do the following:

1. To state the meaning of the sine, cosine, and tangent of an angle.

2. To find the value of the sine, cosine, and tangent of a given acute angle *a.* by measuring.
b. by using the table.

3. To solve problems involving right triangles.

4. To find the sine, cosine, and tangent of an angle which lies between two consecutive angles given in the table.

5. *a.* To find the sine of an angle when the cosine is given.

b. To find the cosine when the sine is given.

c. To find the tangent when either the sine or cosine is given.

39. Typical problems and exercises. The following exercises are types of work that every pupil should be able to do:

1. Draw an angle equal to 58° and find by measurement sin 58°, cos 58°, tan 58°.

2. Find the height of a pole whose shadow is 23 feet long when the angle of elevation of the sun is 58°.

3. Find by interpolation the value of sin 28° 36′; cos 28° 36′.

4. Sin $x = \frac{2}{3}$. Find cos x and tan x.

CHAPTER IV

EXPONENTS

Laws of Exponents

40. The meaning of an exponent. You have learned that symbols like a^2, a^3, a^4, a^n denote *products* in which the same factor is used several times. Thus, $a^2 = a \cdot a$, $a^3 = a \cdot a \cdot a$, $a^n = a \cdot a \cdot a \ \ldots$ to n factors. The number of factors used in a^n depends upon the value of n. If n takes the values 1, 2, 3, etc., a^n has the corresponding meaning a, $a \cdot a$, $a \cdot a \cdot a$, etc.

a^n is called a *power*, a is the *base* of the power, and n is the *exponent.*

The meaning given above to the symbol a^n implies that n is a *positive integer*. It excludes such symbols as a^{-2}, $a^{\frac{1}{3}}$, a^0, $a^{\sqrt{2}}$.

41. The law of exponents for finding the product of two powers having equal bases. The product $a^3 \cdot a^2$ means $(a \cdot a \cdot a)(a \cdot a)$, or $a \cdot a \cdot a \cdot a \cdot a$, or a^5.

$\therefore a^3 \cdot a^2 = a^5$.

Similarly, give the meaning and the final result for each of the following products:

$a^2 \cdot a^4$

Solution: $a^2 \cdot a^4 = (a \cdot a)(a \cdot a \cdot a \cdot a) = a^6$.

$m^2 \cdot m^3$	$5^3 \cdot 5$	$p^4 \cdot p^6$
$t^2 \cdot t^7$	$3^4 \cdot 3^2$	$r \cdot r^{10}$

Careful examination of the results in the exercises above leads to the following law: *When two powers having equal bases are multiplied, the product is a power having the same base as the given powers and an exponent equal to the sum of the exponents of the given powers.*

In symbols, the law is expressed by means of the formula $$\boldsymbol{a^m \cdot a^n = a^{m+n}}.$$

Exercises

Find the following products as indicated:

1. $a^3 \cdot a^2$

Solution: $a^3 \cdot a^2 = a^{3+2} = a^5$

2. $y^7 \cdot y^4$ **7.** $(\frac{1}{2})^2(\frac{1}{2})^6$ **12.** $(6a^2c^2y)(-3a^3cy^2)$

3. $q^4 \cdot q^9$ **8.** $(-2)^4(-2)$ **13.** $(-5x^3yz^4)(8xy^4z^2)$

4. $x^2 \cdot x^3 \cdot x$ **9.** $(\frac{2}{3})^3(\frac{2}{3})^2$ **14.** $x^{2a} \cdot x^{3a}$

5. $q \cdot q^2 \cdot q^9$ **10.** $(-a)^2(-a)^3$ **15.** $a^{3x} \cdot a^{5x}$

6. $s^5 \cdot s \cdot s^3$ **11.** $(-a)(-a)^3(-a)^2$ **16.** $x^{a+b} \cdot x^{a-b}$

Multiply as indicated:

17. $8ab^2 - 3a^2bc^3$ by $5a^3b^3c$.

Suggestion: Multiply each term of the binomial by the monomial.

18. $5x^2y + 3xy^2 + y^3$ by $x^2 - y$.

Suggestion: Multiply every term of the trinomial by each term of the binomial.

19. $m^3 + 3m^2n + 6mn^2 + n^3$ by $m^2 + n^2$.

42. A law for finding the power of a product. The power $(2 \cdot 3)^3$ means $(2 \cdot 3)\ (2 \cdot 3)\ (2 \cdot 3)$,

$$\text{or}\quad 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 3,$$
$$\text{or}\quad 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3,$$
$$\text{or}\quad 2^3 \cdot 3^3.$$
$$\therefore (2 \cdot 3)^3 = 2^3 \cdot 3^3 = 8 \cdot 27 = 216.$$

Similarly find the power of each of the following products as indicated:

$(3\cdot 4)^2$	$(3\cdot 2)^4$	$(ab)^2$
$(2\cdot 5)^2$	$(2\cdot 3\cdot 4)^2$	$(mnt)^4$

The exercises above lead to the following law: *The power of a given product is equal to the product of its factors, each factor having been raised to an exponent equal to that of the given product.*

In symbols this law is expressed briefly by the formula $$(\boldsymbol{abc})^m = \boldsymbol{a}^m\boldsymbol{b}^m\boldsymbol{c}^m.$$

Exercises

In the following raise the products to powers as indicated:

1. $(-3xy)^2$

Solution: $(-3xy)^2 = (-3)^2x^2y^2 = 9x^2y^2$.

2. $(2xy)^2$	6. $(\frac{1}{3}mn)^3$	10. $(ab)^n$
3. $(3abc)^3$	7. $(-\frac{1}{2}nx)^2$	11. $(xy)^{2a}$
4. $(-2x)^2$	8. $(2ab\cdot 3cd)^2$	12. $(2x)^2+(-3y)^2+(-z)^3$
5. $(-xyz)^3$	9. $(3abc)^4$	13. $(3xy)^3-(2xy)^2+(-xy)^4$

43. A law for finding the power of a power. The expression $(3^2)^3$ means $3^2\cdot 3^2\cdot 3^2$, or $3^4\cdot 3^2$, or 3^6.

Thus $(3^2)^3 = 3^6$.

Similarly, $(a^2)^4$ means $a^2\cdot a^2\cdot a^2\cdot a^2 = a^8$.

Thus $(a^2)^4 = a^8$.

Find the following powers as indicated:

$(x^4)^5$	$(a^5)^6$	$(m^4)^4$
$(e^4)^4$	$(m^3)^3$	$(b^2)^3$
$(x^6)^4$	$(s^6)^4$	$(a^5)^5$

The preceding exercises show that a power of a power may be found briefly by the following law: *The power of a given power is a power whose base is the base of the given power and whose exponent is the product of the given exponents.*

In symbols this is expressed by the formula:

$$(a^m)^n = a^{mn}$$

EXERCISES

Apply the law above to the following:

1. $(x^4)^2$

Solution: $(x^4)^2 = x^{4\cdot 2} = x^8$.

2. $(a^5)^2$
3. $(a^4)^3$
4. $(m^4)^8$
5. $(a^3)^7$
6. $(x^2)^{2a}$
7. $(a^{n+1})^2$
8. $(y^{p+q})^{p-q}$
9. $(a^{x+1})^{x+4}$
10. $(x^{m+2})^{m-3}$.

11. $(2a^2b^2)^3$

Solution: $(2a^2b^2)^3 = 2^3(a^2)^3(b^2)^3 = 8a^6b^6$.

12. $(a^2b^3)^5$
13. $(a^3b^4c^2)^5$
14. $(9p^2q^4)^2$
15. $(aq^4)^3$
16. $(x^3yz^5)^2$
17. $(x^2y^{12}z)^4$

44. A law for finding the quotient of two powers having equal bases. To divide a^5 by a^2 we may proceed as in arithmetic,

$$\text{i.e., } a^5 \div a^2 = \frac{a^5}{a^2}$$

$$= \frac{\cancel{a}\cdot\cancel{a}\cdot a\cdot a\cdot a}{\cancel{a}\cdot\cancel{a}.} = a^3.$$

$$\therefore \frac{a^5}{a^2} = a^3.$$

Similarly carry out the following divisions:

$$\frac{x^4}{x^3} \qquad \frac{b^5}{b} \qquad \frac{b^3}{b}$$

$$\frac{a^7}{a^5} \qquad \frac{m^7}{m^3} \qquad \frac{y^4}{y^3}$$

By examining the results in the preceding exercises show that in finding the quotient of two powers having equal bases *the exponent in the quotient is equal to the exponent in the dividend minus the exponent in the divisor.*

In symbols this may be expressed by the formula

$$\frac{\boldsymbol{a}^m}{\boldsymbol{a}^n} = \boldsymbol{a}^{m-n}.$$

Exercises

Divide as indicated:

1. $\frac{a^9}{a^4}$

Solution: $\frac{a^9}{a^4} = a^{9-4} = a^5.$

2. $\frac{m^6}{m^3}$
3. $\frac{m^4}{m^2}$
4. $\frac{y^{10}}{y^6}$
5. $\frac{12a^4}{6a^2}$
6. $\frac{20a^5}{4a^2}$
7. $\frac{a^{3n}}{a^n}$
8. $\frac{b^{m+1}}{b^2}$
9. $\frac{t^n}{t^{n-1}}$
10. $\frac{a^{2r+3}}{a^r}$
11. $\frac{2x^2y^3}{4xy^2}$

Solution: $\dfrac{\overset{x^{2-1}y^{3-2}}{\cancel{2}\cancel{x^2}\cancel{y^3}}}{\underset{2\cdot 1\cdot 1}{\cancel{4}\cancel{x}\cancel{y^2}}} = \dfrac{x^{2-1}y^{3-2}}{2} = \dfrac{xy}{2}.$

12. $\frac{x^2y^3}{x^2y}$

13. $\frac{a^3b^4c^2}{ab^3c}$

14. $\frac{-36x^6y^4z^2}{6x^4y^3z^2}$

15. $\frac{65a^6b^5c^3}{-13ab^4}$

16. $\frac{-96m^4n^3t}{-12m^2n}$

17. $\frac{-96x^8y^{12}}{-16x^3y^5}$

45. A law for finding a power of a quotient. The meaning of $\left(\frac{a}{b}\right)^3$ is $\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{a}{b}\cdot$

Multiplying you have $\frac{a\cdot a\cdot a}{b\cdot b\cdot b}$, or $\frac{a^3}{b^3}\cdot$

$$\text{Thus, } \left(\frac{a}{b}\right)^3=\frac{a^3}{b^3}\cdot$$

Similarly find

$$\left(\frac{a}{b}\right)^4 \qquad \left(\frac{x}{y}\right)^2 \qquad \left(\frac{r}{x}\right)^5$$

It is seen from the results of these exercises that *to raise a quotient to a given power you first raise the dividend and the divisor to the given power, and then divide the first by the second.*

In symbols this may be expressed by the formula

$$\left(\frac{\boldsymbol{a}}{\boldsymbol{b}}\right)^m=\frac{\boldsymbol{a}^m}{\boldsymbol{b}^m}.$$

Exercises

Using the law above change the following powers:

1. $\left(\frac{m}{t}\right)^6$

2. $\left(\frac{x}{y}\right)^4$

3. $\left(\frac{a}{b}\right)^n$

4. $\left(\frac{r}{t}\right)^{k+1}$

5. $\left(\frac{t}{s}\right)^{2a}$

6. $\left(\frac{a}{b}\right)^{2r-1}$

7. $\left(\frac{x^2}{y^3}\right)^5$

Solution: $\left(\frac{x^2}{y^3}\right)^5=\frac{(x^2)^5}{(y^3)^5}=\frac{x^{10}}{y^{15}}.$

8. $\left(\frac{a^5}{b^3}\right)^2$

9. $\left(\frac{m^3}{n^4}\right)^3$

10. $\left(\frac{x^2b^3}{xb^2}\right)^5$

11. $\left(\frac{2x^2y^3}{3a^2b}\right)^4$

12. $\left(\frac{-4b^3x^2}{2b^2y^3}\right)^3$

13. $\left(\frac{4p^4q}{9a^3b^2}\right)^3$

14. $\left(\frac{x^a}{y^b}\right)^2$

15. $\left(\frac{m^2r}{n^3s}\right)^a$

16. $\left(\frac{a^2b^2c}{x^3yz^2}\right)^n$

NEGATIVE AND ZERO EXPONENTS

46. The meaning of a zero exponent. So far in all of your work you have used exponents to indicate the number of equal factors in a product. Thus, x^3 means x taken as a factor 3 times. An exponent, therefore, has always been a *positive whole number.* The following shows that a meaning may be given to a *zero* exponent.

You may divide a^2 by a^2 as in arithmetic, and find the quotient to be 1, *i.e.*, $\frac{a^2}{a^2}=\frac{a\cdot a}{a\cdot a}=1.$

If the law $\frac{a^m}{a^n}=a^{m-n}$ is assumed to hold when $m=n$,

then $\frac{a^2}{a^2}=a^{2-2}=a^0.$

To make the two results for $\frac{a^2}{a^2}$ agree, it is agreed that a^0 should have the value 1. In other words, a^0

may be thought of as indicating that a power of a has been divided by itself.

Show that x^0, $(a+b)^0$, $(2a-3b+c)^0$ all have the same value, *i.e.*, that they are all equal to 1.

In general, $$\boldsymbol{a^0=1}.$$

Exercises

Give the value of each of the following:

1. x^0
2. 5^0
3. $(-18)^0$
4. $(\frac{1}{2})^0$
5. $(x-y)^0$
6. $\left(\frac{x}{y}\right)^0$
7. $5(\frac{3}{4})^0$
8. $\frac{27}{3^0}$
9. $2^2\times4^2\times6^0$
10. $a+b+(a+b)^0$
11. $6(x-y)^0+x-y$
12. $\frac{x^0y^3z^4}{xyz^2}$

47. The meaning of negative exponents may be found as follows: Let it be assumed that the law $\frac{a^m}{a^n}=a^{m-n}$ holds when n is greater than m.

Then $\frac{a^2}{a^5}=a^{2-5}=a^{-3}$.

But the value of $\frac{a^2}{a^5}$ may be found by changing the fraction to the simplest form. This gives

$$\frac{a^2}{a^5}=\frac{\cancel{a}\cdot\cancel{a}}{\cancel{a}\cdot\cancel{a}\cdot a\cdot a\cdot a}=\frac{1}{a^3}$$

To make the two results agree, the symbol a^{-3} is given the meaning $\frac{1}{a^3}$.

Similarly, $a^{-4}=\frac{1}{a^4}$, $a^{-5}=\frac{1}{a^5}$, . . .

In general, $$\boldsymbol{a^{-n}=\frac{1}{a^n}}.$$

Exercises

Give the value of each of the following:

1. 5^{-2}	7. $(-8)^{-2}$	13. $(a+a^{-1})^2$
2. 4^{-3}	8. a^2b^{-3}	14. $a^{-2}\cdot a^8\cdot a^{-1}$
3. $2^{-1}\cdot 3^{-2}$	9. $6a^{-2}xy^{-1}$	15. $(\frac{2}{3})^{-1}$
4. $6\cdot 4^{-2}$	10. $3(-x)^{-2}y$	16. $(\frac{4}{5})^{-2}$
5. $2^{-1}\cdot 8^0\cdot 3$	11. $a^{-1}+b^{-1}$	17. $\frac{1}{3^{-4}}$
6. $6^2\cdot 4^{-3}\cdot 3^0$	12. $(2a)^{-2}x^2$	18. $\frac{a}{b^{-2}}$

Raising a Binomial to a Power

48. A law for finding the power of a binomial. You have learned by multiplying that

$$(a+b)^2=a^2+2ab+b^2$$

and $(a+b)^3=a^3+3a^2b+3ab^2+b^3$.

Multiply $(a+b)^3$ by $(a+b)$ and show the result to be

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

Similarly,

$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$$

The four examples above enable you to find a law by which the power of a binomial can be found directly without first multiplying. By examining the four equations above you find:

1. The exponent of a in the first term of each polynomial is the same as that of the binomial. Hence, you can always find the first term by inspection.

2. The exponent of a in the succeeding terms decreases by 1 with each term, being zero in the last term.

3. The exponent of b is zero in the first term and increases by 1 with each succeeding term.

4. For each term the coefficient may be obtained from the preceding term as follows: State the coefficient of the preceding term, multiply it by the exponent of a in that term, and divide the result by the number of that term.

5. The number of terms is one more than the exponent of the binomial.

Sir Isaac Newton

Sir Isaac Newton (1642–1727) won his greatest fame because he discovered the law of gravitation. He wrote on algebra and on the theory of equations. He extended the binomial theorem to include not only the case when the exponent has positive integral values but also when it has fractional and negative values.

At his time Newton was considered the world's greatest leader in the subjects of mathematics and physics.

Show that you may find the sixth power of $(a+b)$ as follows:

$$(a+b)^6=a^6+6a^5b+\frac{\overset{3}{\cancel{6}}\cdot 5}{\cancel{2}}a^4b^2+\frac{\cancel{3}\cdot 5\cdot 4}{\cancel{3}}a^3b^3+\frac{5\cdot\cancel{4}\cdot 3}{\cancel{4}}a^2b^4+\dots$$

$$=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+\dots$$

This law of finding the power of a binomial is called the *binomial theorem.*

Exercises

Find without multiplying the polynomials equal to the following products:

1. $(x+y)^8$
2. $(3a+1)^5$
3. $(2a+b)^5$
4. $(2a-b)^5$
5. $\left(3m-\frac{n}{2}\right)^5$

Solution:

$$\left(3m-\frac{n}{2}\right)^5=(3m)^5-5(3m)^4\left(\frac{n}{2}\right)+\frac{5\cdot\overset{2}{\cancel{4}}}{\cancel{2}}(3m)^3\left(\frac{n}{2}\right)^2$$

$$-\frac{10\cdot\cancel{3}}{\cancel{3}}(3m)^2\left(\frac{n}{2}\right)^3+\frac{\overset{5}{\cancel{10}}\cdot\cancel{2}}{\underset{2}{\underset{\cancel{2}}{\cancel{4}}}}(3m)\left(\frac{n}{2}\right)^4-\frac{\cancel{5}\cdot 1}{\cancel{5}}\left(\frac{n}{2}\right)^5$$

$$=243m^5-\frac{5\cdot 3^4m^4n}{2}+\frac{10\cdot 3^3m^3n^2}{2^2}$$

$$-\frac{10\cdot 3^2m^2n^3}{8}+\frac{5\cdot 3mn^4}{16}-\frac{n^5}{32}$$

$$=243m^5-\frac{405m^4n}{2}+\frac{135}{2}m^3n^2-\frac{45}{4}m^2n^3+\frac{15}{16}mn^4-\frac{n^5}{32}.$$

6. $(\frac{3}{4}x-\frac{1}{2}y)^4$
7. $(\frac{1}{2}a+2b)^4$
8. $(3a+\frac{1}{2})^4$
9. $(3x-2y)^5$
10. $\left(2-\frac{a}{3}\right)^4$
11. $\left(3a^2-\frac{b}{2}\right)^4$

12. $(2x^2+3ay)^5$

13. $(a^2-b^2)^6$

14. $(3x^2-\frac{1}{2})^4$

15. $(5a+\frac{1}{5})^4$

49. A formula for finding a required term in the expansion of $(a+b)^n$ without finding the other terms.* If only one of the terms of the expansion of $(a+b)^n$ is needed, it will save time and effort if you do not have to write out the whole expansion. A formula for the rth term is found as follows:

You know that $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2$

$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \ldots$$

To state the rth term, you must find the coefficient, the exponent of a, and the exponent of b.

Tabulating the numerator of the coefficient you have:

Number of term	*Numerator of term*
3	$n(n-1)$
4	$n(n-1)(n-2)$
5	$n(n-1)(n-2)(n-3)$
6	$n(n-1)(n-2)(n-3)(n-4)$
,	
,	
,	
$\therefore r$	$n(n-1)(n-2)(n-3)\ (n-4)\ \ldots\ (n-[r-2])$

Similarly, the denominator, the exponent of a, and the exponent of b are as follows:

* §49 may be omitted or assigned as a supplementary topic.

Number of term	*Denominator of term*	*Exponent of* **a**	*Exponent of* **b**
3	$1\cdot 2$	$n-2$	2
4	$1\cdot 2\cdot 3$	$n-3$	3
5	$1\cdot 2\cdot 3\cdot 4$	$n-4$	4
6	$1\cdot 2\cdot 3\cdot 4\cdot 5$	$n-5$	5
.			
.			
.			
$\therefore r$	$1\cdot 2\cdot 3\cdot 4\cdot 5\ \ldots\ r-1$	$n-(r-1)$	$r-1$

Hence the rth term is

$$\frac{n(n-1)(n-2)\ \ldots\ (n-r+2)}{1\cdot 2\cdot 3\cdot\ \ldots\ (r-1)}a^{n-r+1}b^{r-1}$$

Find the following required terms:

1. 5th term of $(m+x)^7$

Solution: the 5th term $=\dfrac{7\cdot\cancel{6}\cdot 5\cdot\cancel{4}}{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}}m^{7-5+1}x^{5-1}=35m^3x^4$

2. 4th term of $(x-y)^8$

3. 5th term of $(2+a)^7$

4. 6th term of $(1+a)^{16}$

5. 7th term of $(a-b)^8$

6. 6th term of $(2a-3b)^9$

7. 4th term of $(2x^2-\frac{1}{3})^8$

8. The amount a of a principal of p dollars, invested for n years, at a rate r, interest compounded annually, is given by the formula $a=p(1+r)^n$. At the end of 4 years what will be the amount of \$100 invested at 5% compounded annually?

9. Find the amount of \$350 at compound interest at 3% for 6 years.

10. Find the amount of \$25 at compound interest for 5 years at 5%.

11. How much money must be deposited at compound interest at 4% in order to have an amount equal to \$100 in 5 years?

12. If interest is compounded several (m) times a year the amount is given by the formula

$$a=p\left(1+\frac{r}{m}\right)^{mn}$$

A principal of \$250 is invested at interest of 5% compounded semi-annually. What will be the amount at the end of 4 years?

An Important Law for Deriving a Series of Numbers

50. How to form a geometric progression. The series of numbers 1, 4, 16, 64, . . . is a *geometric progression.* Each term is obtained by multiplying the preceding term by 4. Similarly, the series 4, 8, 16, 32, . . . and a, a^2, a^3, a^4, . . . are geometric progressions because each term is found by multiplying the preceding term by a fixed number.

As in the case of the arithmetical progression (§9) and the binomial expansion (§49), a formula is to be derived for finding any particular term in the series without finding others.

51. A formula for finding a required term of a geometric progression. Arranging the number of the term and the corresponding term of the series $a+ar+ar^2+ar^3+$ . . . in tabular form you have

Number of term	1	2	3	4	5	6	. . .	n
Term	a	ar	ar^2	ar^3	ar^4	ar^5	. . .	ar^{n-1}

Note that for each term the exponent of r is 1 less than the number of the term. Hence, the nth term is ar^{n-1}. This gives the formula

$$\boldsymbol{l=ar^{n-1}},$$

where n is the number of the term, l is the nth term, a the first term, and r the constant multiplier.

Exercises

By means of the formula find the required term in the following exercises:

1. The 10th term of $3+1+\frac{1}{3}+\ \ldots$

2. The 8th term of $1-\frac{1}{2}+\frac{1}{4}-\ \ldots$

3. The 12th term of $\frac{4}{3}+\frac{2}{3}+\frac{1}{3}+\ \ldots$

4. The 6th term of $-2-\frac{1}{4}-\frac{1}{32}-\ \ldots$

5. The 15th term of $p+p(1+r)+p(1+r)^2+\ \ldots$

6. A rubber ball falls from a height of 20 feet. On each rebound it rises 40% of the previous height. How far does it fall on the seventh descent?

7. By means of the formula $l=ar^{n-1}$ prove the formula $a=p(1+r)^n$ which was used in Exercise 8 (§49).

Suggestion: Show that the amounts for the first, second, third years, . . . are $p(1+r)$, $p(1+r)^2$, $p(1+r)^3$, . . .

52. A formula for finding the sum of a geometric progression. To work out this formula let the sum of n terms be given:

$$s=a+ar+ar^2+ar^3+\ \ldots\ +ar^{n-2}+ar^{n-1}$$

Multiply both members of the equation by r.

Then $rs=ar+ar^2+ar^3+ar^4\ \ldots\ +ar^{n-1}+ar^n$.

Subtracting you have

$$s - rs = a - ar^n.$$

$$\therefore s(1-r) = a - ar^n.$$

$$\therefore s = \frac{a - ar^n}{1-r}.$$

EXERCISES

1. A chain letter is sent to two persons with the request that each send two copies to 2 friends requesting that they send 2 copies each to 2 friends, etc. When 8 sets of letters have been sent, how many copies have been made?

2. A man plans to save each year one half as much again as during the preceding year. He begins by saving \$114 the first year. What will be the total amount of his savings at the end of the 10th year?

3. John's father gave him \$1 on his 12th birthday and promised to double the gift each birthday up to the 21st with the agreement that all the money be put in John's savings account. How much money will he save during that time?

4. Each stroke of a pump withdraws one fourth of the air remaining in a vessel. After 8 strokes what part of the original amount remains?

5. A man deposits \$100 each year for 15 years, receiving 4% interest compounded annually. How much will he save in the 15 years?

Suggestion: Use the formula $a = p(1+r)^n$ to find the amount accumulated at each payment. These amounts form a geometric progression. Then use the formula for finding the sum of n terms of a geometric progression. A simple method of finding the value of this sum will be shown in Chapter VI.

53. What every pupil should be able to do. Having studied Chapter IV you should be able to do the following:

1. To use correctly each of the laws of exponents taught in the Chapter.

2. To work problems containing negative or zero exponents.

3. To find the polynomial equal to a power of a binomial by using the binomial theorem, and to find any required term in the expansion by means of the formula.

4. To use the two formulas of the geometric progression in the solution of problems.

54. Typical problems and exercises. The following indicate the type and difficulty of problems and exercises which every pupil should be able to work out:

1. Simplify: $(x^5y^2z)^6$; $(-c^2d^3)^6$; $\left(\frac{x^2}{y^3}\right)^4$; a^2b^{-5}; $(3x)^{-2}y^2$; $2^{-3}m^2n^{-4}$; $(a^{-2})^4$; $x^7 \cdot x^2 \cdot x^0$; $\frac{8a^4b^0}{2ab^2}$; $\frac{3x^2y^2}{c^{-2}}$.

2. Find the first 5 terms of $(x^2-3a)^7$.

3. Find, without expanding, the 4th term of $(2a+b)^6$.

4. Using the formula $a=p(1+r)^n$ find the amount of $225 invested at compound interest at 4% at the end of the third year.

5. Using the formula find the 10th term of the series

$$1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\ . \ . \ .$$

6. If you put on your savings account $1.00 the 10th birthday, twice as much on the 11th, etc., for 8 birthdays, show by means of the formula the total amount saved.

CHAPTER V

FRACTIONS—FACTORING

What You Are Going to Study in This Chapter

55. Why you must study fractions. In your previous work problems were usually solved by adding, subtracting, multiplying, and dividing whole numbers (integers). However, there are other problems which lead to polynomials or equations containing fractions. Knowledge of fractions is therefore necessary to solve them. The problem below illustrates this:

An automobile concern has taken a contract for 126 automobiles. One of the factories of the company can make that many automobiles in 15 days, and another in 10 days. The question arises as to how many days it would take to fill the contract if both factories worked together.

To answer the question denote the number of days in which both factories together can make the 126 automobiles by n.

Then $\dfrac{126}{n}$ is the number they make in 1 day.

However, the first factory makes $\frac{1}{15}$ of 126 in one day, and the second factory makes $\frac{1}{10}$ of 126.

Hence, together they make $\frac{126}{15}+\frac{126}{10}$ in one day.

It follows that

$$\frac{126}{n}=\frac{126}{15}+\frac{126}{10}.$$

This is a fractional equation containing n in the denominator.

In Chapter V you will learn enough about fractions to enable you to solve this and other fractional equations.

You know from arithmetic that one has to be able to perform the following operations with fractions:

1. To change fractions to the simplest form.
2. To multiply fractions.
3. To divide fractions.
4. To add and subtract fractions.

All four processes are as essential in algebraic work as in arithmetical work. The first is especially important because in all operations the final result should generally be changed to the simplest form.

In changing an *arithmetical* fraction, as $\frac{126}{15}$, into the simplest form, and in adding fractions, as in finding such sums as $\frac{126}{15}+\frac{126}{10}$, it is necessary to find the divisors (factors) of numbers. Certain principles have been found helpful in discovering them. They enable us to tell by inspection whether such numbers as 2, 3, and 5 are factors of a number. You must know them thoroughly if you are to find quickly the factors common to numerator and denominator of a fraction, or the least common multiple of the denominators of several fractions which are to be added. Similarly, in *algebraic* work you must be familiar with the principles which help you to find the factors of polynomials which occur in algebraic

fractions. Hence, factoring of polynomials is to be another important part of the work of this chapter.

Changing Fractions to Lowest Terms

56. How fractions are changed to lowest terms. If in the fraction $\frac{6}{15}$ the numerator 6 and the denominator 15 are divided by the factor 3, which is contained in both, the fraction is said to be *changed to lowest terms.* To *change a fraction to lowest terms* means to divide numerator and denominator by the largest factor contained in both. It involves two steps:

1. *You must factor the numerator and the denominator.*

2. *You must divide the numerator and denominator by factors common to both.*

For example, to change the fraction $\frac{2ac-3bc}{2ab-3b^2}$ to lowest terms,

(1) change numerator and denominator to factored forms $\frac{c(2a-3b)}{b(2a-3b)}$.

(2) divide numerator and denominator by the factor $2a-3b$.

This changes the fraction to $\frac{c}{b}$.

Briefly, the work may be arranged as follows:

$$\frac{2ac-3bc}{2ab-3b^2}=\frac{c\overset{1}{\cancel{(2a-3b)}}}{b\underset{1}{\cancel{(2a-3b)}}}=\frac{c}{b}.$$

In dividing the numerator $c(2a-3b)$ by $2a-3b$ the following principle has been applied: *Dividing a factor of a product by a number divides the product by that number.*

You must keep in mind that the quotient found by dividing $2a-3b$ by itself is 1. To save time, the 1 is usually not written, but it should be understood.

Exercises

Change each of the following fractions to the simplest form:

1. $\dfrac{4(x+y)}{12(x+y)}$

Suggestions: Divide the numerator and denominator by 4; divide the numerator and denominator by $x+y$.

2. $\dfrac{7(a-b)}{14(a-b)}$

3. $\dfrac{4(m+n)^2}{2(m+n)}$

4. $\dfrac{3x^2yz(a+b)}{2xyz(a+b)}$

5. $\dfrac{12a^2b}{18a^2b^2}$

6. $\dfrac{45a^4bc^3}{63a^3b^2c^5}$

7. $\dfrac{45xy^4z^2}{20y^5z^3}$

8. $\dfrac{36m^2n^2}{54m^3n^4k}$

9. $\dfrac{5(a+2)(a+7)}{15(a-2)(a+7)}$

10. $\dfrac{2x^2-2y^2}{5(x^2-y^2)}$

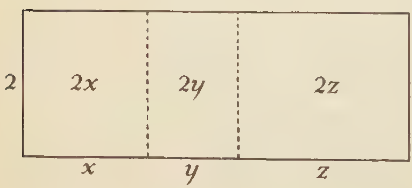


Fig. 20

57. Factoring a polynomial whose terms have a common factor. If all the terms of a polynomial, as

$$2x+2y+2z,$$

contain the same factor, then you may say:

1. This factor is a factor of the polynomial.
2. The other factor is obtained by dividing each term of the polynomial by the common factor.

Thus, 2 is one factor of $2x+2y+2z$, and the other is $x+y+z$. Why?

Geometrically this means that, if the area of a given rectangle (Fig. 20) is $2x+2y+2z$, one side is 2 and the other $x+y+z$.

Similarly, in the polynomial $x^3(x+1)-1(x+1)$ you see that the common factor is $x+1$. By dividing each of the two terms by $x+1$, the other factor is found to be x^3-1.

Hence, $x^3(x+1)-1(x+1)=(x+1)(x^3-1)$.

Exercises

Factor the following, and in each case check the results by multiplying mentally one factor by the other:

1. $15x^2+25x+40$.

Solution: The greatest factor contained in all terms is 5. Dividing by 5, you find the other factor to be

$$3x^2+5x+8.$$

$$\therefore 15x^2+25x+40=5(3x^2+5x+8).$$

2. y^2+2y.

Suggestions: The greatest factor common to both terms is y. By dividing you find the other factor to be $y+2$.

3. $8m^2+24m$.
4. a^2b+ab^2.
5. $6xy^2-3x$.
6. $4ab^2-3a^2b$.
7. $16x^2y^2+48x^2y-16xy+16x$.
8. $4x^2y^2-6x^2y^3+12x^4y^2z^2$.
9. $x^5+x^4+x^3+x^2+x$.
10. $14a^7-49a^5+21a^3-7a$.

Change each of the following fractions to simplest form:

11. $\dfrac{3ab+3b^2}{5a^2+5ab}$.

Solution: First, factor numerator and denominator. This gives

$$\frac{3ab+3b^2}{5a^2+5ab}=\frac{3b(a+b)}{5a(a+b)}.$$

Then divide numerator and denominator by $a+b$, which gives the result $\frac{3b}{5a}$.

Hence $\frac{3ab+3b^2}{5a^2+5ab}=\frac{3b\cancel{(a+b)}}{5a\cancel{(a+b)}}=\frac{3b}{5a}$.

12. $\frac{2a^2b-2a^2}{3a^3b-3a^3}$.

13. $\frac{x^2y^2+xy}{(xy+1)^2}$.

14. $\frac{5a^2-5ab}{5ax+5ay}$.

15. $\frac{5x^2y-5xy-60y}{3ax^2-3ax-36a}$.

Multiplying Fractions

58. Multiplication of arithmetical fractions. To multiply two fractions, as $\frac{2}{5}$ and $\frac{3}{7}$, the product of the numerators is divided by the product of the denominators.

Thus $\frac{2}{5}\times\frac{3}{7}=\frac{2\times 3}{5\times 7}=\frac{6}{35}$.

If in multiplying fractions the resulting product is not in the simplest form, the change should be made before the multiplication is carried out.

$$\text{Thus } \frac{2}{5}\times\frac{10}{6}=\frac{\overset{1}{\cancel{2}}\times\overset{2}{\cancel{10}}}{\underset{1}{\cancel{5}}\times\underset{3}{\cancel{6}}}=\frac{2}{3}.$$

Hence, in the multiplication of *arithmetical fractions* the following two steps are involved:

1. *Multiply the numerators, multiply the denominators, and write the first product over the second.*

In the example above, this gives $\frac{2\times 10}{5\times 6}$.

2. *Change the resulting fraction to simplest form and multiply the remaining factors.* This gives $\frac{2}{3}$.

59. Multiplication of algebraic fractions. The algebraic operation is the same as that of multiplying arithmetical fractions. The following example illustrates it:

Multiply: $\dfrac{3a-3b}{5c-5d}\times\dfrac{10ac-10ad}{6a-6b}.$

Solution: 1. Multiply numerator by numerator and denominator by denominator:

$$\frac{(3a-3b)(10ac-10ad)}{(5c-5d)(6a-6b)}.$$

2. Factor numerator and denominator:

$$\frac{3(a-b)10a(c-d)}{5(c-d)6(a-b)}.$$

3. Divide numerator and denominator by factors common to both:

$$\frac{\cancel{3}\cancel{(a-b)}\overset{2}{\cancel{10}}a\cancel{(c-d)}}{\cancel{5}\cancel{(c-d)}\underset{2}{\cancel{6}}\cancel{(a-b)}}=a.$$

Usually the work should be arranged as follows:

$$\frac{3a-3b}{5c-5d}\times\frac{10ac-10ad}{6a-6b}=\frac{(3a-3b)(10ac-10ad)}{(5c-5d)(6a-6b)}$$

$$=\frac{\cancel{3}\cancel{(a-b)}\overset{2}{\cancel{10}}a\cancel{(c-d)}}{\cancel{5}\cancel{(c-d)}\underset{2}{\cancel{6}}\cancel{(a-b)}}=a.$$

Exercises

Multiply the following:

1. $\dfrac{6x+6y}{2ax-2ay}\times\dfrac{x-y}{x+y}$.
2. $\dfrac{(a+2)(a+3)}{2a+8}\times\dfrac{a+4}{a+2}$.
3. $\dfrac{2\pi r^2+2\pi rh}{x-y}\times\dfrac{3hx-3hy}{\pi r^2}$.
4. $\dfrac{(x+2)(x-3)}{(x+3)(x-1)}\times\dfrac{x+3}{x-3}$.
5. $\dfrac{3a-6b}{4a+2b}\times\dfrac{6(2a+b)^2}{2(a-2b)}$.
6. $\dfrac{x^4-x^2}{x+2}\times\dfrac{9(x+2)^2}{3x^2-3}$.

60. Factoring the difference of two squares. Multiply $a+b$ by $a-b$. The product is a^2-b^2. Similarly, if the sum of any two numbers is multiplied by the difference of the *same* two numbers, the product is the difference of the squares of the two numbers.

Find the product of each of the following, first by inspection, then verify the result by multiplication:

$$(x+y)(x-y) \qquad (a+3)(a-3)$$
$$(2x-5)(2x+5) \qquad (x+\tfrac{1}{2})(x-\tfrac{1}{2})$$

You have seen that a^2-b^2 is the result of multiplying $(a+b)$ by $(a-b)$. Hence, the factors of the difference of two squares may be found by inspection as follows:

1. Find the square root of each square and add the two. This gives the first factor $a+b$.

2. To find the second factor, subtract b from a. This gives $a-b$.

Similarly, to find the factors of x^4-9, find first the square roots of x^4 and 9. They are x^2 and 3. Hence, the factors are the sum of x^2 and 3, and the difference of x^2 and 3, *i.e.*, x^2+3 and x^2-3.

Exercises

Factor each of the following binomials and test the correctness of your factors by multiplying them mentally:

1. $16a^6-9b^2$.

Solution: The square roots of $16a^6$ and $9b^2$ are $4a^3$ and $3b$. The sum is $4a^3+3b$ and the difference is $4a^3-3b$.

$$\therefore 16a^6-9b^2=(4a^3+3b)(4a^3-3b).$$

Check: $(4a^3+3b)(4a^3-3b)=16a^6+12a^3b-12a^3b-9b^2$
$=16a^6-9b^2$.

2. x^2-25.

3. m^2-36.

4. $16-a^2$.

5. v^2-1.

6. v^4-1.

7. $v^2-\frac{1}{4}$.

8. $1-121x^2$.

9. $\dfrac{1}{a^2}-\dfrac{1}{b^2}$.

10. $64x^2-25$.

11. $v^4t^2-s^2$.

12. $a^6-\frac{1}{4}m^2$.

13. $x^4-\dfrac{4y^2}{9}$.

14. $.25x^2-4$.

15. $a^2-.01b^2$.

16. $\pi r^2-\pi x^2$.

17. a^4-b^4.

18. $16aw^2-at^2$.

19. $(a+b)^2-c^2$.

20. $(m-n)^2-16$.

21. $a^2-(b-c)^2$.

22. $(x+3y)^2-(m+2)^2$.

23. $(3b-2c)^2-(a+t)^2$.

24. $(2x-1)^2-(3b+2)^2$.

25. $(3a-c)^2-(b-d)^2$.

Change each of the following fractions to lowest terms (§56) by dividing numerator and denominator by the same factor:

26. $\dfrac{a^2-1}{(a-1)^2}$.

27. $\dfrac{a^2-b^2}{(a-b)^3}$.

28. $\dfrac{x^2-25}{3x-15}$.

29. $\dfrac{a^2-64}{am+8m}$.

30. $\dfrac{x^2-4y^2}{(x-2y)^2}$.

31. $\dfrac{9-x^2}{(3-x)^2}$.

32. $\dfrac{(m-n)a-(m-n)b}{a^2-b^2}$.

33. $\dfrac{(a+b)(a+c)}{(a^2-b^2)(a^2-c^2)}$.

34. $\dfrac{(a+b)^2-c^2}{2a+2b+2c}$.

35. $\dfrac{(4x-t)^2-(2y-3z)^2}{(4x-2y)^2-(3z-t)^2}$.

Multiply as indicated and change each result to the simplest form:

36. $\dfrac{(x+8)^2}{(x-4)^2} \cdot \dfrac{x^2-16}{x^2-64}$

38. $\dfrac{ab}{a+b} \cdot \dfrac{6a^2b-4ab^2}{45a^2-20b^2} \cdot \dfrac{3a+2b}{4a^2b^2}$

37. $\dfrac{(x-2)^2}{x^2+x} \cdot \dfrac{x^2-1}{x^2-4}$

39. $\dfrac{(3-2a)^2}{2a} \cdot \dfrac{3+2a}{3-2a} \cdot \dfrac{8a^2b}{9-4a^2}$

To save time and labor in computation the following formulas are stated in factored form. Show that the formulas are true.

40. The area of a circular ring (Fig. 21) is $\pi(R+r)(R-r)$.

41. The difference between the areas of two squares (Fig. 22) is $(x+y)(x-y)$.

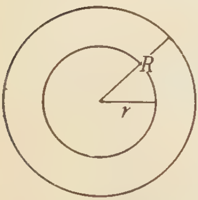


FIG. 21

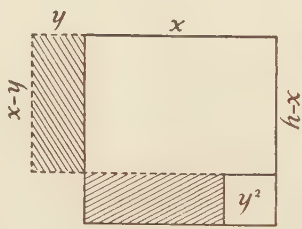


FIG. 22

FIG. 23

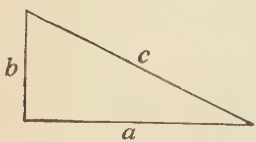


FIG. 24

42. The total area of a right circular cylinder (Fig. 23) is $2\pi R(R+H)$.

43. The side a of a right triangle (Fig. 24) is $\sqrt{(c+b)(c-b)}$.

DIVIDING FRACTIONS

61. Dividing by an arithmetical fraction. You know from arithmetic that when a number, as 25, is to be divided by a fraction, as $\frac{2}{3}$, the divisor is inverted,

and the number is then multiplied by the inverted divisor.

For example, $25 \div \frac{2}{3} = 25 \times \frac{3}{2} = \frac{25 \times 3}{2} = \frac{75}{2}$.

The complete process involves three steps:

1. *The divisor is inverted.*
2. *The dividend is multiplied by the inverted divisor.*
3. *The result is changed to simplest form.*

$$\text{Thus, } \frac{4}{9} \div \frac{2}{3} = \frac{4}{9} \times \frac{3}{2} = \frac{\overset{2}{\cancel{4}} \times \overset{1}{\cancel{3}}}{\underset{3}{\cancel{9}} \times \underset{1}{\cancel{2}}} = \frac{2}{3}.$$

$$\text{And } \frac{6}{10} \div \frac{3}{15} = \frac{6}{10} \times \frac{15}{3} = \frac{\overset{\cancel{2}}{\cancel{6}} \times \overset{3}{\cancel{15}}}{\underset{\cancel{2}}{\cancel{10}} \times \cancel{3}} = 3.$$

62. Dividing by an algebraic fraction. The process of dividing by an algebraic fraction is the same as that of dividing by an arithmetical fraction. Each of the following three examples illustrates the three steps:

1. $$\frac{6x}{3y} \div \frac{8x}{15y^2} = \frac{6x}{3y} \times \frac{15y^2}{8x} = \frac{\overset{\cancel{2}}{\cancel{6x}} \cdot 15\overset{y}{\cancel{y^2}}}{\underset{1}{\cancel{3y}} \cdot \underset{4}{\cancel{8x}}} = \frac{15y}{4}.$$

2. $$\frac{x^2+2x}{x^2-2x} \div \frac{(x+2)^2}{(x-2)^2} = \frac{x^2+2x}{x^2-2x} \cdot \frac{(x-2)^2}{(x+2)^2}$$

$$= \frac{\cancel{x}\cancel{(x+2)}\,(x-2)^{\cancel{2}}}{\cancel{x}\cancel{(x-2)}\,(x+2)^{\cancel{2}}} = \frac{x-2}{x+2}.$$

3. $$\frac{m^2a+ma^2}{m^2b+b^3}\div\frac{3m+3a}{m^4-b^4}=\frac{(m^2a+ma^2)(m^4-b^4)}{(m^2b+b^3)(3m+3a)}$$
$$=\frac{ma(m+a)(m^2+b^2)(m^2-b^2)}{b(m^2+b^2)3(m+a)}=\frac{ma(m^2-b^2)}{3b}.$$

Exercises

Divide as indicated:

1. $\dfrac{(x+3)^2}{x^2-9}\div\dfrac{x+3}{x-3}.$

2. $\dfrac{(x+4y)^2}{x^2+2xy}\div\dfrac{x^2+4xy}{(x+5y)(x+2y)}.$

3. $\dfrac{a^4-b^4}{a^3-ab^2}\div\dfrac{a^2-b^2}{a^2+b^2}.$

4. $\dfrac{a^2-121}{a^2-4}\div\dfrac{a+11}{a+2}.$

5. $\dfrac{4x^2-9y^2}{x^2-4}\div\dfrac{2x-3y}{x-2}.$

6. $\dfrac{4x^2-25y^2}{16x^2-9y^2}\div\dfrac{2xy-5y^2}{4x^2+3xy}.$

7. $\dfrac{a^2b+ab^2}{a^2b+b^3}\div\dfrac{5ab(a+b)^2}{a^4-b^4}.$

8. $\dfrac{xy^2+x^2y}{x^3+xy^2}\div\dfrac{2xy(x-y)}{x^4-y^4}.$

63. Factoring trinomials by trial. If you multiply two binomials, as $2x+3$ and $5x+7$, the work may be arranged as follows:

$$\begin{array}{r}2x+3\\ 5x+7\\ \hline 10x^2+15x\\ +14x+21\\ \hline 10x^2+29x+21\end{array}$$

Thus the product $10x^2+29x+21$ is a quadratic trinomial whose factors are $2x+3$ and $5x+7$.

Notice that the coefficient of x^2 is the product of 2 and 5, that the constant 21 is the product of 3 and 7, and that the coefficient of x is the *sum* of the two cross products 2×7 and 3×5.

Suppose that the trinomial is now given and that the factors are to be found. This means that you must know how to determine the 2 and 3 of the factor $2x+3$ and the 5 and 7 of $5x+7$. This may be done as follows:

Write down two factors, as 2 and 5, of the coefficient of x^2, and two factors of the constant term, as 3 and 7.

$$\boxed{\begin{matrix}2 & & 3\\ & \times & \\ 5 & & 7\end{matrix}}$$

If the sum of the cross products is 29, the required numbers have been found. If not, other factor pairs of 10 and 21 have to be tried. The following examples illustrate further how to find the factors of a quadratic trinomial:

1. Factor $6x^2+7x+2$.

Two factors of 6 are either 6 and 1 or 2 and 3. The factors of 2 are 2 and 1 or 1 and 2. By trial select:

$$\boxed{\begin{matrix}2 & & 1\\ & \times & \\ 3 & & 2\end{matrix}}$$

Since the sum of the cross products is 7, it follows that $6x^2+7x+2=(2x+1)(3x+2)$.

2. Factor $9r^2-6r-8$.

Two factors of 9 are 3 and 3 or 9 and 1. Two factors of 8 are 8 and 1 or 1 and 8; 2 and 4 or 4 and 2. By trial select:

$$\boxed{\begin{matrix}3 & & 2\\ & \times & \\ 3 & & -4\end{matrix}}$$

Since the sum of the cross products is to be -6, a minus sign must be prefixed to the 4.

Hence, $9r^2-6r-8=(3r+2)(3r-4)$.

3. Factor $112m^2+49m^4+64$.

In this trinomial it is necessary to rearrange the terms, which gives $49m^4+112m^2+64$.

By trial select the numbers:

$$\boxed{\begin{matrix}7 & & 8\\ & \times & \\ 7 & & 8\end{matrix}}$$

Hence, $$49m^4+112m^2+64=(7m^2+8)(7m^2+8)$$
$$=(7m^2+8)^2.$$

Exercises

Factor each of the following:

1. $2x^2-3x-5$.
2. $6a^2-7a-5$.
3. $10x^2-13x-30$.
4. $64a^2+16ab+b^2$.
5. $4x^4-8x^2+4$.
6. $6x^2+17x+7$.
7. $2x^2-3x-2$.
8. $35x^2-39x-36$.
9. $5x^2+18xy+16y^2$.
10. $9x^6-6x^3-8$.
11. $x^4y^2+2x^2yz+z^2$.
12. $1-6xy+5x^2y^2$.
13. $a^2b^2-9ab-22$.
14. $10x^2-7xy-12y^2$.
15. $10r^2+3rs-18s^2$.
16. $27x^2-42xy-24y^2$.
17. $16y^2+5y^4+3$.
18. $4x^2+20x+25$.

Divide as indicated:

19. $\dfrac{3b^2+10bx+3x^2}{-3b^2+10bx-3x^2}\div\dfrac{(3b+x)x}{(3b-x)^2}$.

20. $\dfrac{x^2-4}{x^3+3x^2}\div\dfrac{x^2-4x+4}{x^3-x^2-12x}$.

21. $\dfrac{x^2-5xy-14y^2}{x^2+5xy-24y^2}\div\dfrac{x^2-3xy-28y^2}{x^2-8xy+15y^2}$.

22. $\dfrac{a^2b+10ab+21b}{a^3-4a^2+3a} \div \dfrac{a^2b^3-9b^3}{a^3-a^2}$.

23. $\dfrac{x^2-7x+12}{x-1} \div \dfrac{x^2-16}{x^2-1}$.

64. Factoring the sum or difference of two cubes. Multiplying $a+b$ by a^2-ab+b^2, you have

$$\begin{aligned}(a+b)(a^2-ab+b^2) &= a^3-a^2b+ab^2+a^2b-ab^2+b^3\\ &= a^3+b^3.\end{aligned}$$

Similarly,

$$\begin{aligned}(a-b)(a^2+ab+b^2) &= a^3+a^2b+ab^2-a^2b-ab^2-b^3\\ &= a^3-b^3.\end{aligned}$$

The binomials a^3+b^3 and a^3-b^3 are called the *sum of two cubes* and the *difference of two cubes.*

We are interested in the problem of finding the factors of binomials of the form of the sum or of the difference of two cubes. The multiplication above shows that the factors of a^3+b^3 are $a+b$ and a^2-ab+b^2, and that the factors of a^3-b^3 are $a-b$ and a^2+ab+b^2.

The following states how to find the factors of binomials of these two forms:

1. To find the first factor, $a+b$, of a^3+b^3, take the cube root of each of the terms a^3 and b^3, and then add them.

2. Then derive the second factor from the first factor by squaring the first term, subtracting the product of the two terms, and adding the square of the second term.

For example, to find the factors of $27a^3+64$, first note that this is the sum of two cubes.

To find the first factor, take the cube root of $27a^3$, the cube root of 64, and add the results. This gives $3a+4$.

To find the second factor, square the $3a$, subtract the product $(3a)(4)$, and add the square of 4. This gives $9a^2-12a+16$.

Hence, $27a^3+64=(3a+4)(9a^2-12a+16)$.

To find the factors of the difference of two cubes, as $27x^3-64$, proceed similarly as follows:

1. Find the cube roots of $27x^3$ and 64, and take the difference. This gives the factor $3x-4$.

2. Square the first term of this factor, add the product of the two terms, and add the square of the second term.

Hence, $27x^3-64=(3x-4)(9x^2+12x+16)$.

Note the following: When a binomial is the *sum* of two cubes, the *sum* of the cube roots is taken as the first factor; in the second factor their product is *subtracted.*

For the *difference* of two cubes, the *difference* is taken as the first factor, and the product in the second factor is *added.*

Exercises

Factor each of the following, and test the results by multiplying the factors mentally:

1. a^3+27.	**7.** $1-64a^6$.	**13.** $40x^3-5$.
2. x^3-64.	**8.** x^6-y^6.	**14.** x^4-8a^3x.
3. x^3-64y^3.	**9.** $x^3+8y^3z^3$.	**15.** a^9-b^9.
4. $8x^3-1$.	**10.** $512a^3-b^6$.	**16.** $\frac{x^3}{27}-y^3$.
5. $y^3+\frac{1}{8}$.	**11.** a^6+b^6.	**17.** $a^3b^6c^9+8d^6$.
6. $3x^3+24$.	**12.** $.008a^3-b^3$.	**18.** $(x+y)^3-27$.

Change the following fractions to lowest terms:

19. $\dfrac{x^6-y^6}{x^4-y^4}$.

21. $\dfrac{4a^2b^2+1}{64a^6b^6+1}$.

20. $\dfrac{x^3-1}{x^2+x+1}$.

22. $\dfrac{m^2+mn}{m^3+n^3}$.

Multiply and divide as indicated:

23. $\dfrac{a^6-b^6}{(a-b)^2}\div\dfrac{a^2+ab+b^2}{a-b}$.

24. $\dfrac{y^2-9}{y^3-27}\div\dfrac{y^2+3y+9}{y+3}$.

25. $\dfrac{a^3-3ab}{a^3-b^3}\div\dfrac{a^2-10ab+21b^2}{a^2+ab+b^2}$.

26. $\dfrac{a^3+8b^3}{a^3-8b^3}\cdot\dfrac{a-2b}{a+2b}\cdot\dfrac{a^2+2ab+4b^2}{a^2-2ab+4b^2}$.

Adding and Subtracting Fractions

65. How fractions are added or subtracted in arithmetic. Since the operation of adding and subtracting algebraic fractions is the same as that for arithmetical fractions, it will be shown first what is done when fractions are added in arithmetic.

The simplest case is that of adding fractions having the same denominators, as $\frac{5}{8}+\frac{1}{8}$. The problem is not different from that of adding 5 marbles and 1 marble, or 5 oranges and 1 orange, or 5 dollars and 1 dollar.

Thus, as 5 marbles + 1 marble = 6 marbles,
and as 5 dollars + 1 dollar = 6 dollars,
so you have $\frac{5}{8}+\frac{1}{8}=\frac{6}{8}$.

Briefly, *to add fractions having the same denominator, first add the numerators, and then divide the result by the common denominator.*

66. Addition and subtraction of algebraic fractions having the same denominator. By the method of §65 the fractions $\frac{5}{a}$ and $\frac{1}{a}$ may be added by adding the numerators 5 and 1 and dividing the sum by the denominator a, *i.e.*,

$$\frac{5}{a}+\frac{1}{a}=\frac{6}{a}.$$

EXERCISES

Perform the following additions and subtractions orally:

1. $\frac{2}{x}+\frac{8}{x}.$

2. $\frac{6}{y}-\frac{2}{y}.$

3. $\frac{a}{b}+\frac{c}{b}.$

4. $\frac{a}{c}+\frac{b}{c}.$

5. $\frac{3}{a+1}+\frac{4}{a+1}.$

6. $\frac{a}{m+n}+\frac{b}{m+n}.$

7. $\frac{x}{a+b}+\frac{x-y}{a+b}.$

8. $\frac{m}{x+y}-\frac{m-n}{x+y}.$

9. $\frac{a}{a-1}-\frac{ab}{a-1}.$

10. $\frac{x}{x+y}+\frac{y}{x+y}.$

11. $\frac{2a-b}{x}-\frac{a+2b}{x}.$

12. $\frac{5+3c}{a+b}-\frac{1-c}{a+b}.$

67. Addition and subtraction of algebraic fractions having different denominators. As in arithmetic, fractions with different denominators must be changed to fractions having the same denominator before they can be combined.

To illustrate: to add $\frac{3}{7}$ and $\frac{5}{14}$, change $\frac{3}{7}$ to $\frac{6}{14}$ and then combine $\frac{6}{14}$ and $\frac{5}{14}$.

Thus, $\frac{3}{7}+\frac{5}{14}=\frac{6}{14}+\frac{5}{14}=\frac{11}{14}$.

Similarly, $\dfrac{a}{x}+\dfrac{b}{2x}=\dfrac{2a}{2x}+\dfrac{b}{2x}=\dfrac{2a+b}{2x}$.

The following examples further illustrate the process:

1. $\dfrac{1}{a^3}+\dfrac{1}{a^2}+\dfrac{1}{a}$.

Note that a^3 is the least common denominator to be used. Hence, multiply numerator and denominator of $\dfrac{1}{a^2}$ by a and of $\dfrac{1}{a}$ by a^2.

Thus, $\dfrac{1}{a^3}+\dfrac{1}{a^2}+\dfrac{1}{a}=\dfrac{1}{a^3}+\dfrac{a}{a^3}+\dfrac{a^2}{a^3}$.

Adding the numerators, you have the result $\dfrac{1+a+a^2}{a^3}$.

2. $\dfrac{1}{4}+\dfrac{2x}{3y}-\dfrac{5}{xy}$.

Solution: The least common denominator is $12xy$.

$$\text{Hence, } \frac{1}{4}+\frac{2x}{3y}-\frac{5}{xy}=\frac{1\cdot 3xy}{12xy}+\frac{2x\cdot 4x}{12xy}-\frac{5\cdot 12}{12xy}$$
$$=\frac{3xy+8x^2-60}{12xy}.$$

3. $x+\dfrac{3}{y}+\dfrac{1}{xy}$.

Solution: Consider the term x as a fraction with denominator equal to 1.

Then $x+\frac{3}{y}+\frac{1}{xy}=\frac{x}{1}+\frac{3}{y}+\frac{1}{xy}$

$$=\frac{x\cdot xy}{xy}+\frac{3\cdot x}{xy}+\frac{1}{xy}=\frac{x^2y+3x+1}{xy}.$$

EXERCISES

Add and subtract as indicated:

1. $\frac{a}{5b}-\frac{2}{b}$.

2. $\frac{1}{a}-\frac{1}{5a}$.

3. $\frac{3}{4}+\frac{x}{x+1}$

4. $8+\frac{1}{a}$.

5. $\frac{a^2}{b^2}+1$.

6. $\frac{(x-y)^2}{2xy}-1$.

7. $\frac{a}{a-1}-\frac{ab}{a(a-1)}$.

8. $\frac{4}{a+2}+\frac{3}{a-2}$.

9. $\frac{3m}{m+2}+\frac{4}{m-2}$.

10. $\frac{6}{x^2-9}+\frac{2}{x+3}$.

11. $\frac{2x}{3y-2x}+\frac{x}{2x-3y}$.

Suggestion: Multiply numerator and denominator of $\frac{x}{2x-3y}$ by -1.

12. $\frac{1}{2a-3b}+\frac{a+b}{4a^2-6ab}$.

13. $\frac{3}{a^2+5a+6}-\frac{a}{a+2}$.

14. $\frac{1}{x-y}+\frac{1}{x+y}+\frac{1}{x^2-y^2}$.

15. $\frac{3}{5x}+\frac{2x}{x-y}-\frac{2y}{x+y}$.

16. $\frac{x}{x^2-1}+\frac{x+3}{x-1}-\frac{x-3}{x+1}$.

17. $\frac{2}{5a+10b}-\frac{7}{3a+6b}+\frac{9}{2a+4b}$

18. $\frac{2}{x^2-10x+21}+\frac{1}{x-7}+\frac{2}{x-3}$.

19. $\frac{5x}{x+y}+\frac{xy}{x^2-y^2}-\frac{4y}{y-x}$.

Suggestion: Multiply numerator and denominator of $\frac{4y}{y-x}$ by -1.

20. $a-3+\frac{a^3-27}{a^2+3a+9}$.

21. $\frac{2x^2-3y^2}{27x^3-64y^3}-\frac{2x+y}{9x^2-16y^2}$.

22. $\frac{1}{m^2-4m-5}-\frac{1}{m^2-6m+5}$.

23. $\dfrac{x^2-y^2}{(x+y)^2}+\dfrac{x-y}{x+y}-\dfrac{x^2+y^2}{x^2-y^2}.$

24. $\dfrac{2x+3}{x-6}-\dfrac{x^2-11x+18}{x^2-36}-\dfrac{x-6}{x+6}.$

68. Factoring polynomials by grouping terms. You have previously (§57) learned how to factor polynomials whose terms contain a common factor. The following three examples illustrate another method of factoring polynomials. Work each one in full:

1. Let $ax+ay+bx+by$ be the polynomial to be factored.

Notice that the first two terms can be factored. Hence, consider them as a group. Similarly, consider the last two terms as one group.

Thus $ax+ay+bx+by=\overparen{ax+ay}+\overparen{bx+by}.$
Factoring each group gives $a(x+y)+b(x+y).$

Since $x+y$ is now a factor common to both terms, you have $a(x+y)+b(x-y)$ equal to $(x+y)(a+b)$ (§57).

The work may now be arranged briefly:

$\overparen{ax+ay}+\overparen{bx+by}=a(x+y)+b(x+y)$, by factoring each group $=(x+y)\ (a+b).$

2. $\overparen{x^4+x^3}-\overparen{x-1}=x^3(x+1)-1(x+1)$, by factoring
$=(x+1)\ (x^3-1)$ each group by
$=(x+1)\ (x-1)\ (x^2+x+1).$ §57

3. $\overparen{a^4-\overparen{a^2b^2+b^2}-1}=(a^2+1)\ (a^2-1)-b^2(a^2-1)$
$=(a^2-1)\ (a^2+1-b^2)$
$=(a+1)\ (a-1)\ (a^2+1-b^2).$

Exercises

Factor each of the following polynomials:

1. $6y^3-15y^2-8y+20$.
2. x^3+x^2+x+1.
3. $3ax-ay+9bx-3by$.
4. $a^3-b^3+a^2-b^2$.
5. $x-y+x^3-y^3$.
6. $x^3-3x^2+4x-12$.
7. x^2-9y^2+x+3y.
8. $x^4+x^3y-xy^3-y^4$.
9. $ax^2-2a^2x-x+2a$.
10. $ax+ay+bx+by-cx-cy$.
11. $x^2y+y^2z+xz^2-x^2z-xy^2-yz^2$.
12. $x^3+xz^2+x^2y+xy^2+y^3+yz^2$.

Reduce the following to lowest terms:

13. $\dfrac{y^2+ay+by+ab}{(y^2-a^2)(y^2-b^2)}$.

14. $\dfrac{x^2-px-qx+pq}{x^2-px-rx+pr}$.

15. $\dfrac{2mx-6x-2my+6y}{3mnx-9nx-3mny+9ny}$.

16. $\dfrac{m^3-mn^2-m^2+n^2}{(m^4-n^4)(m^2-2m+1)}$.

Each of the following polynomials can be changed into the difference of two squares. Exercises 17, 18, and 19 illustrate the method.

17. $x^2-6x+9-y^2$.

Solution: Grouping the first three terms you have

$$\begin{aligned}\overbrace{x^2-6x+9}-y^2&=(x-3)^2-y^2\\&=(x-3+y)\ (x-3-y)\end{aligned}$$

18. $a^2-x^2-y^2+2xy$.

Solution: Grouping the last three terms you have

$$\begin{aligned}a^2-x^2-y^2+2xy&=a^2-(x^2-2xy+y^2).\\&=a^2-(x-y)^2\\&=(a+x-y)(a-x+y).\end{aligned}$$

19. $m^2+6m-x^2+9-4xy-4y^2$.

Solution: Grouping the first, second, and fourth terms you have

$$\begin{aligned}&\overparen{m^2+6m+9}-(x^2+4xy+4y^2)\\&=(m+3)^2-(x+2y)^2\\&=(m+3+x+2y)\ (m+3-x-2y).\end{aligned}$$

Factor the following:

20. $x^2+2xy+y^2-z^2$.

21. $z^2-a^2-b^2-2ab$.

22. $9x^2-4a^2-4ab-b^2$.

23. $12ab+25-4a^2-9b^2$.

24. $a^4-12a^2+36-x^2$.

25. $x^4-x^2+12xy-36y^2$.

The following polynomials can be changed to trinomials. Exercise 26 illustrates the method:

26. $a^2+b^2+2ab+8a+8b-9$.

Solution: Grouping the first three terms and the fourth and fifth, you have the trinomial

$$\begin{aligned}&\overparen{a^2+2ab+b^2}+\overparen{8a+8b}-9\\&=(a+b)^2+8(a+b)-9\\&=(a+b+9)\ (a+b-1).\end{aligned}$$

Factor the following:

27. $x^2+2xy+y^2+4mx+4my+4m^2$.

28. $x^2+6xy+9y^2+5x+15y+6$.

29. $4x^2+4xy+y^2-8x-4y-21$.

30. $a^2-6ab+9b^2-4a+12b-12$.

Factor by grouping each of the following:

31. x^3+y^3+x+y.

32. $a^3-b^3-(a-b)^2$.

33. $x^3-a^3-2(x-a)$.

34. $a^3+b^3-(a+b)$.

69. Factoring a quadratic trinomial by completing the square. The trinomial $x^2+x^2y^2+y^4$ cannot be factored by the trial method (§63). It is almost a

perfect square. This suggests that you add the term x^2y^2 to it to form the perfect square $x^4+2x^2y^2+y^4$. This, of course, is no longer equal to the original trinomial, but by subtracting the number just added you have the equality

$$x^4+x^2y^2+y^4=x^4+2x^2y^2+y^4-x^2y^2,$$

which may be written $(x^2+y^2)^2-x^2y^2$.

Since this is the difference of two squares, the factors are (x^2+y^2+xy) and (x^2+y^2-xy).

The work may now be summarized as follows:

$$\begin{aligned} x^4+x^2y^2+y^4 &= x^4+2x^2y^2+y^4-x^2y^2 \\ &= (x^2+y^2)^2-x^2y^2 \\ &= (x^2+y^2+xy)\ (x^2+y^2-xy). \end{aligned}$$

Exercises

Factor the following:

1. x^4+x^2+1.
2. $9a^4-15a^2+1$.
3. $x^4-3x^2y^2+y^4$.
4. $4m^4-13m^2+1$.
5. $4z^4-29z^2+25$.
6. $a^4+2a^2b^2+9b^4$.
7. $25x^4-31x^2y^2+16y^4$.
8. x^4+4.

Change to lowest terms:

9. $\dfrac{a^4+a^2b^2+b^4}{a^2+ab+b^2}$.
10. $\dfrac{c^4+c^2y^2+y^4}{(c-y)(c^2+cy+y^2)}\div\dfrac{y^2+c^2}{c^2-y^2}$.

70. Factoring polynomials having a factor of the form $x\pm a$.* You have learned (§26) that when a polynomial $f(x)$ is divided by $x-a$ the remainder is $f(a)$; that $f(a)$ can be found by synthetic division; and that the same process determines the quotient.

*§70 may be omitted or assigned to special pupils as supplementary work.

To illustrate: let us divide $f(x)=x^3-2x^2-9x+20$ by $x-2$. The process is as follows:

$$\begin{array}{lrrrr|l} & 1 & -2 & -9 & 20 & \underline{\ 2\ } \\ & & 2 & 0 & 18 & \\ \hline \text{Quotient} = & 1 & 0 & -9 & \boxed{2} & = \text{Remainder} = f(2). \end{array}$$

$$\text{Thus, } \frac{x^3-2x^2-9x+20}{x-2}=x^2-9+\frac{2}{x-2}.$$

If the remainder happens to be zero, the division is exact, $x-2$ is a factor, and the quotient is a second factor.

For example, let $f(x)=x^3-2x^2-9x+18$.

Dividing synthetically by 2, you have

$$\begin{array}{rrrr|l} 1 & -2 & -9 & 18 & \underline{\ 2\ } \\ & 2 & 0 & -18 & \\ \hline 1 & 0 & -9 & \boxed{0} & \end{array}$$

Since the remainder is zero, you have

$$\begin{aligned} x^3-2x^2-9x+18 &= (x-2)(x^2-9) \\ &= (x-2)(x+3)(x-3). \end{aligned}$$

Hence, to find factors of the form $x+a$, proceed as follows: *Divide the polynomial synthetically, using as trial divisors the factors of the constant term. If the remainder is zero,* **x** *minus the trial number is a factor.*

Exercises

Factor the following polynomials:

1. $x^3+x^2-14x-24$.

Solution: The trial divisors are ±1, ±2, ±3, ±4, ±6, ±8, ±12, ±24.

Beginning with 1, you have

$$\begin{array}{rrrr|l} 1 & 1 & -14 & -24 & \underline{\ 1\ } \\ & 1 & 2 & -12 & \\ \hline 1 & 2 & -12 & \boxed{-36} & \end{array}$$

$$\begin{array}{rrrr|l} 1 & 1 & -14 & -24 & \underline{-1} \\ & -1 & 0 & 14 & \\ \hline 1 & 0 & -14 & \boxed{-10} & \end{array}$$

$$\begin{array}{rrrr|l} 1 & 1 & -14 & -24 & \underline{\ 2\ } \\ & 2 & 6 & -16 & \\ \hline 1 & 3 & -8 & \boxed{-40} & \end{array}$$

$$\begin{array}{rrrr|l} 1 & 1 & -14 & -24 & \underline{-2} \\ & -2 & 2 & 24 & \\ \hline 1 & -1 & -12 & \boxed{0} & \end{array}$$

$\therefore x+2$ and x^2-x-12 are factors,

i.e., $x^3+x^2-14x-24=(x+2)(x^2-x-12)$
$=(x+2)(x-4)(x+3)$.

2. $x^3-9x^2+26x-24$.

3. $x^3-3x^2-10x+24$.

4. $x^3+9x^2+27x+27$.

5. $x^3-6x^2+11x-6$.

6. $x^3-6x^2+12x-8$.

7. x^4+x^3-x-1.

8. $y^4+4y^3-8y-32$.

9. x^4-5x^2+4.

71. Summary of factoring: The methods of factoring taught in this chapter may be summarized as shown below. This summary should be helpful to you when trying to decide what method to apply to a particular problem:

1. *Binomials.* When a binomial is to be factored, it should be examined to see whether it is the difference of two squares, as x^2-y^2, the difference of two cubes,

as x^3-y^3, or the sum of two cubes, as x^3+y^3. The methods to be used are those of §§60 and 64.

2. *Trinomials.* In factoring trinomials of the form ax^2+bx+c, use the trial method of §63. If the trial method does not work, it may be possible to complete the square using the method of §69.

3. *Polynomials.* If the polynomial contains a factor common to all terms, use the method of §57. When a polynomial has no common factor and contains more than 3 terms use the method of grouping, §68, or see whether it contains factors of the form $x \pm a$ (§70).

Exercises

Factor the following:

1. $40x^3-5$.
2. x^4-16.
3. $a^4-4x^2y^2$.
4. $1728x^3-1$.
5. a^4-8ab^3.
6. $6x^2+11x-10$.
7. $9x^2-24x+16$.
8. x^4-2x^2+1.
9. $x^4+3x^2y^2-4y^2$.
10. $p^2-x^2-2xy-y^2$.
11. $a^2-2ab+b^2-y^2$.
12. $1-5x^2+4x^4$.
13. $ac-bc-ad+bd$.
14. a^3+a-b^3-b.
15. $\frac{x^2}{y^2}+2-\frac{3y^2}{x^2}$.
16. $(a^2+b^2)^2-16x^2y^2$.
17. $a^3+a-x-x^3$.
18. $a^3-c^3-a^2+c^2$.
19. p^3-3p^2+3p-1.
20. x^2-9y^2+x+3y.
21. $3x^3-3x+4x^4-4x^2$.
22. x^3+2x^2-4x-8.
23. x^3-7x+6.
24. $x(x+1)(4x-5)-6(x+1)$.
25. $a^4+a^2b^2+b^2-1$.
26. $(y^2-5y)^2-2(y^2-5y)-24$.
27. $(x+y)^2-(x^2-y^2)-6(x-y)^2$.
28. x^5-1 (use §70).
29. a^4+4 (use §69).
30. $32-a^5$; a^5-b^5.
31. a^7-b^7; x^6+y^6.

72. Supplementary exercises on fractions.* The following exercises are difficult and complex. Before attempting to carry out the operations, study each carefully and select the best method of work. When there are brackets containing parentheses, it is usually best to perform the operations in the parentheses first.

Exercises

1. $$\left[\frac{1}{mx}\left(\frac{m}{x}+\frac{x}{m}\right)\div\left(\frac{m^6+x^6}{m^3\cdot x^3}\right)\right]\left(m^2-x^2+\frac{x^4}{m^2}\right)\left(\frac{am+mx}{cx-ax}\right).$$

Solution:

$$=\left[\frac{1}{mx}\left(\frac{m^2+x^2}{mx}\right)\div\frac{m^6+x^6}{m^3x^3}\right]\frac{m^4-m^2x^2+x^4}{m^2}\cdot\frac{am+mx}{cx-ax}$$

$$=\frac{(m^2+x^2)\cdot m^3x^3\cdot(m^4-m^2x^2+x^4)(am+mx)}{mx\cdot mx(m^6+x^6)\cdot m^2\cdot(cx-ax)}$$

$$=\frac{\cancel{(m^2+x^2)}\ \cancel{m^3}\cancel{x^3}\ \cancel{(m^4-m^2x^2+x^4)}\ \cancel{m}(a+x)}{\cancel{mx}\cdot\cancel{mx}\ \cancel{(m^2+x^2)}\cancel{(m^4-m^2x^2+x^4)}\cancel{m^2}\cancel{x}(c-a)}$$

$$=\frac{a+x}{c-a}.$$

2. $$\left(\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}\right)\left(1+\frac{2x}{y-x}\right)\div\left(\frac{y}{x}-\frac{x}{y}\right).$$

3. $$\left(2a-1+\frac{6a-11}{a+4}\right)\div\left(a+3-\frac{3a+17}{a+4}\right).$$

4. $$\left(x-1+\frac{5}{x+1}\right)\left(\frac{x^2}{6}+\frac{x}{4}+\frac{1}{12}\right)\div\left(\frac{2x^3+8x}{3x+3}\right).$$

5. $$\left[\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)\div\left(\frac{x}{y}+\frac{y}{x}\right)\right]\left[y^2-\frac{y^4}{y^2-x^2}\right].$$

*May be omitted.

6. $\left(a^2+\dfrac{b^4}{a^2-b^2}\right)\left(a^2+b^2\right)\div\left(\dfrac{a}{a+b}+\dfrac{b}{a-b}\right).$

7. $\dfrac{8c^3-1}{9c^2-12c+4}\left(1-\dfrac{4}{3c+2}\right)\div\left(\dfrac{2c-1}{9c^2-4}\right).$

8. $\left[\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}+\dfrac{4b^2}{b^2-a^2}\right]\div\left[\dfrac{a-b}{a+b}-1\right].$

9. $\dfrac{r}{(r-s)(s-t)}+\dfrac{s}{(s-r)(s-t)}+\dfrac{t}{(t-r)(t-s)}.$

Suggestion:

$$=\frac{r}{(r-s)\ (r-t)}+\frac{-s}{\underset{(r-s)}{\cancel{(s-r)}}\,(s-t)}+\frac{t}{\underset{(r-t)\ (s-t)}{\cancel{(t-r)}\,\cancel{(t-s)}}}$$

$$=\frac{r(s-t)-s(r-t)+t(r-s)}{(r-s)(r-t)(s-t)},\ \text{etc.}$$

10. $\dfrac{x+y}{(x-z)(y-z)}-\dfrac{y+z}{(x-y)(z-x)}-\dfrac{x+z}{(z-y)(y-x)}.$

11. $\dfrac{\dfrac{a-1}{a}}{a-\dfrac{1}{a}}.$

Suggestion: 1. Perform the division as indicated, *i.e.*, subtract $\frac{1}{a}$ from a, invert the resulting fraction and multiply.

2. Obtain the result more easily by multiplying numerator and denominator by the least common denominator a. Thus,

$$\frac{\dfrac{a-1}{a}}{a-\dfrac{1}{a}}=\frac{\dfrac{a(a-1)}{a}}{a^2-\dfrac{a}{a}}=\frac{a-1}{a^2-1}=\frac{1}{a+1}.$$

12. $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{b}-\frac{1}{a}}$. 13. $\dfrac{p+\frac{a}{b}}{q-\frac{c}{d}}$. 14. $\dfrac{\frac{x+3y}{m-5n}}{\frac{x-3y}{m+5n}}$. 15. $\dfrac{\frac{a}{\frac{a}{b}+1}}{\frac{b}{\frac{a}{b}-1}}$.

The Use of Factoring in Solving Equations

73. Solving equations by factoring. Let it be required to solve the equation $x^2-16=0$.

Factoring the left member you have

$$(x+4)(x-4)=0.$$

This equation is satisfied if $x+4=0$, or $x-4=0$, or if both factors are zero. Thus, the value of x which makes a factor equal to zero, also satisfies the original equation $x^2-16=0$.

To find this value, put a factor equal to zero, as

$$x+4=0.$$

It follows that $x=-4$.

Hence, $x_1=-4$ satisfies the original equation.

Similarly, from $x-4=0$, you have the root

$$x_2=4.$$

The preceding solution of the equation $x^2-16=0$ may now be briefly arranged as follows:

Given: $x^2-16=0$.

By factoring, $(x+4)(x-4)=0$.

Let $x+4=0$

and $x-4=0$.

Then $x_1=-4$

and $x_2=4$.

Exercises

Solve the following equations by factoring:

1. $4x^2=9.$

Solution:

$$4x^2=9.$$

Subtracting 9, you have $4x^2-9=0.$

By factoring, $(2x+3)(2x-3)=0.$

Let $2x+3=0.$

Then $x_1=-\frac{3}{2}.$

Let $2x-3=0.$

Then $x_2=\frac{3}{2}.$

Check for $x_1=-\frac{3}{2}$:

Left Member		Right Member
$4(-\frac{3}{2})^2$		9
$4(\frac{9}{4})$		9
9	=	9

Check for $x_2=+\frac{3}{2}$:

Left Member		Right Member
$4(\frac{3}{2})^2$		9
$4(\frac{9}{4})$		9
9	=	9

2. $16y^2=25.$

3. $225a^2-9=0.$

4. $49x^2=16.$

5. $9x^2=4.$

6. $y^2-25=0.$

7. $2a^2-50=0.$

8. $x^2+6x=0.$

Solution: $x^2+6x=0.$

$$x(x+6)=0.$$

$$\therefore x_1=0.$$

$$x+6=0.$$

$$\therefore x_2=-6.$$

9. $2y^2-10y=0$.

10. $3x^2=5x$.

11. $d^2-7d=0$.

12. $24r^2-2r=0$.

13. $x^2+\frac{5x}{2}=0$.

14. $3x^2+7x=0$.

15. $x^2-13x+12=0$.

Solution: $x^2-13x+12=0$.

$$(x-12)(x-1)=0.$$
$$x-12=0.$$
$$x-1=0.$$
$$\therefore x_1=12$$
and $x_2=1$.

16. $x^2-11x+30=0$.

17. $x^2+45=14x$.

18. $x^2-8x+15=0$.

19. $x^2-4x-21=0$.

20. $a^2=6+a$.

21. $9y^2-12y+4=0$.

22. $x^2+8x-48=0$.

23. $3-x^2=-2x$.

24. $81x^2+18x+1=0$.

25. $50x+24=25x^2$.

26. $9x^2+3x-2=0$.

27. $12x-28=-x^2$.

28. $2x^3+7x^2-3x-18=0$.

Solution: Factoring the left member, you have

$$(x+2)(x+3)(2x-3)=0.$$

Put $x+2=0$.

$$x+3=0.$$
$$2x-3=0.$$

Then $x_1=-2$.

$$x_2=-3.$$
$$x_3=\tfrac{3}{2}.$$

29. $y^3-3y-2=0$.

30. $x^3-3x+2=0$.

31. $y^3+2y^2-y-2=0$.

74. What every pupil should be able to do. In Chapter V you have been taught to do the following:

1. To add, subtract, multiply, and divide fractions.
2. To change fractions to lowest terms.
3. To factor polynomials of the following forms:

a. Polynomials having a factor common to all terms.

b. The difference of two squares.

c. The difference or sum of two cubes.

d. Trinomials of the form ax^2+bx+c.

e. Polynomials which by grouping can be changed into the preceding forms.

f. Polynomials containing factors of the form $x \pm a$.

4. To solve equations by the method of factoring.

75. Typical problems and exercises. The following indicate the type of exercises you should be able to work out:

Factor each of the following:

1. $15xy^2-3x$.

2. $a^2-\frac{1}{16}$.

3. $(a+b)^2-9t^2$.

4. $9a^2-2a-7$.

5. $3a^2+18ab+24b^2$.

6. $27a^3-1$; a^6+b^6.

7. y^3+x^3+y+x.

8. $ay^2-3a^2y-y+3a$.

9. $a^4+4a^2x^2+16x^4$.

10. $a^3+3a^2-10a+24$.

11. a^4-8a^2+7.

12. p^3-5p^2+5p-1.

Change each of the following fractions to the simplest form:

13. $\dfrac{3x^2-3xy}{3xm+3xn}$.

14. $\dfrac{a^2-36}{am+6m}$.

15. $\dfrac{3ab+1}{27a^3b^3+1}$.

16. $\dfrac{5a^2b-5ab-60b}{8ma^2-8ma-96m}$.

Multiply:

17. $\dfrac{a^4-a^2}{a+2}\times\dfrac{9(a+2)^2}{3a^2-3}$.

18. $\dfrac{(a-2)^2}{a^2+a}\cdot\dfrac{a^2-1}{a^2-4}$.

Divide:

19. $\dfrac{x^2-121}{x^2-4}\div\dfrac{x+11}{x+2}$.

20. $\dfrac{a^2-7a+12}{a-1}\div\dfrac{a^2-16}{a^2-1}$.

21. $\dfrac{y^2-9}{y^3-27}\div\dfrac{y^2+3y+9}{y+3}$.

Add and subtract:

22. $\dfrac{6}{a^2-9}+\dfrac{2}{a+3}$.

23. $\dfrac{3}{a^2+5a+6}-\dfrac{a}{a+2}$.

24. $x+3+\dfrac{x^3-27}{x^2-3x+9}$.

Solve the following equations:

25. $2a^2-50=0$.

26. $x^2-4x-21=0$.

27. $9a^2-12a+4=0$.

28. $y^3-3y+2=0$.

CHAPTER VI

THE USE OF LOGARITHMS

The Meaning of Logarithms

76. How to use a table to multiply numbers. Tables are used because they save time and labor in numerical computation. They enable you to change the processes of multiplication and division to the simpler processes of addition and subtraction, and to find powers and roots of numbers by performing very simple exercises in multiplication and division.

The following shows how a table of exponents may be used to multiply two numbers:

Arrange in the form of a table a series of numbers obtained by raising 2 to various powers (Fig. 25) as shown in the middle column. By multiplication you have $2^1=2$, $2^2=4$, $2^3=8$, etc. Write the numbers equal to the powers of 2 in the first column, and the exponents in the third.

(1) N	(2) Powers	(3) Exponents
2	2^1	1
4	2^2	2
8	2^3	3
16	2^4	4
32	2^5	5
64	2^6	6
128	2^7	7
256	2^8	8
512	2^9	9
1024	2^{10}	10
2048	2^{11}	11
4096	2^{12}	12

Fig. 25

Use the table to multiply 16 by 64 as follows:

In the column headed N locate 16 and to the right find 2^4.

Likewise, in column N locate 64 and to the right find 2^6.

Note that $16 \times 64 = 2^4 \times 2^6 = 2^{10}$.

Locating 2^{10} in column (2) find, to the left in column (1), the number 1024. This is the required product. To simplify the work repeat what has been said but use column (3) instead of column (2) as shown below:

In column (1) locate 16 and, passing to column (3), find 4.

In column (1) locate 64 and, passing to column (3), find 6.

Add: $4+6=10$.

In column (3) locate 10 and, in column (1), find 1024. This is the required product.

Only one arithmetical operation has been used in finding the product, the addition of 4 and 6. The table has done the rest. The table will have to be made much more complete before it can be used to multiply any two given numbers, but it illustrates the method.

Exercises

Using the table (Fig. 25) find the following products:

1. 32×128.

2. 16×256.

3. 1024×4.

4. 128×8.

5. 512×4.

6. 16×32.

77. How to use a table to divide numbers. Fig. 25 may be used to divide one number by another. For example, to divide 4096 by 256, locate 4096 in column (1) and to the right find 2^{12} in column (2).

Then locate 256 in column (1) and to the right find 2^8 in column (2).

Since $\dfrac{4096}{256}=\dfrac{2^{12}}{2^8}=2^{12-8}=2^4$,

locate 2^4 in column (2) and, to the left, find 16.

The required quotient is 16.

The only arithmetical operation used is the subtraction of 8 from 12.

As above in multiplying, the quotient may be found in a simple way from columns (1) and (3). The steps are as follows:

Locate 4096 in column (1) and, to the right, find 12.
Locate 256 in column (1) and, to the right, find 8.
Subtracting 8 from 12 you have 4.
Locate 4 in column (3) and, to the left, find 16.
This is the required quotient.

Exercises

By means of the table (Fig. 25) find the following quotients:

1. $4096 \div 64$.

2. $128 \div 32$.

3. $512 \div 128$.

4. $256 \div 64$.

5. $2048 \div 32$.

6. $1024 \div 256$.

78. What is meant by a logarithm. The *exponents* in column (3) (Fig. 25) are called *logarithms*. The table is a *logarithmic table*. Thus, 8 is the logarithm which corresponds to the number 256. Briefly, we say that "the logarithm of 256 is 8, if 2 is used as base." Tables could be constructed with other numbers as bases just as the table of Fig. 25 was formed with the base equal to 2.

In general, it is said that *the logarithm of a number N is the exponent to which a base must be raised to give a power equal to N.*

Thus, since $2^8=256$, $\log_2 256=8$. This is read "the logarithm of 256 to the base 2 is 8."

The statements $2^8=256$ and $\log_2 256=8$ really give the same information in different forms.

Exercises

Using the table of Fig. 25 find the logarithms of the numbers below. State your results as shown in Exercise 1.

1. 128.

Solution: $\log_2 128=7$.

Check: $2^7=128$.

2. 256.	**5.** 64.	**8.** 4096.
3. 1024.	**6.** 2.	**9.** 4.
4. 8.	**7.** 16.	**10.** 512.

Common Logarithms

N	Powers	L
100,000	10^5	5
10,000	10^4	4
1,000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
.1	10^{-1}	-1
.01	10^{-2}	-2
.001	10^{-3}	-3
.0001	10^{-4}	-4

Fig. 26

79. A table of logarithms to the base 10. The table of logarithms (Fig. 25) was made by using 2 as a base. In the tables which are generally used for computation the base is 10.

Proceeding as in §76 verify the table in Fig. 26. In this table the first column contains the numbers, the second the powers of 10 equal to the

numbers, and the third the exponents, or logarithms.

A logarithmic table whose base is 10 is called a table of *common logarithms.*

The base 10 is usually not written. It is understood that any base other than 10 must always be indicated. Thus, $\log_{10} 100=2$, is written more simply $\log 100=2$.

Exercises

Find the logarithms of the following numbers:

1. 10,000.	**3.** $\frac{1}{10}$.	**5.** 100,000.
2. 1.	**4.** $\frac{1}{100}$.	**6.** $\frac{1}{1000}$.

7. Find the logarithm of $\frac{1}{3}$ to the base 3.

Solution: Since $3^{-1}=\frac{1}{3}$, it follows that $\log_3 \frac{1}{3}=-1$.

Find the following logarithms and give the reason for each:

8. $\log_4 16$.	**12.** $\log_5 25$.	**16.** $\log_3 81$.
9. $\log_3 9$.	**13.** $\log_5 625$.	**17.** $\log_3 729$.
10. $\log_6 216$.	**14.** $\log_7 49$.	**18.** $\log_4 64$.
11. $\log_3 243$.	**15.** $\log_8 512$.	**19.** $\log_6 36$.

20. Simplify $\log_2 4+\log_3 81+\log_4 64$.

21. Simplify $\log_2 64-\log_3 9+\log_2 1$.

80. Logarithms of numbers not contained in Fig. 26. The table in Fig. 26 contains only a few logarithms. However, it gives some information about the logarithms of numbers which do not appear in the table. You can see that the logarithms of all numbers between 10 and 100 are greater than 1 and less than 2. It follows that the logarithm of 58 must be 1 plus a fraction. Likewise, the logarithm of 258 must be 2 plus a fraction because 258 lies between 100 and 1000.

All logarithms, including those in the table, consist of two parts, the integral (whole) part and the fractional part. The fractional part is always expressed as a decimal. For the numbers in the table (Fig. 26) the fractional part is zero.

The logarithm of 246.5 lies between 2 and 3, since 246.5 lies between 100 and 1000. Therefore, the integral part of the logarithm is 2. The fractional part is given in special tables (pp. 126 and 127).

The fractional part of log 246.5, expressed to 4 places of decimals, is .3918. Hence, log 246.5 = 2.3918.

The integral part of a logarithm is determined by a simple rule which will be established below.

81. Special names to denote the parts of a logarithm. The integral part of a logarithm is called *characteristic*, the fractional part *mantissa*.

Exercises

The following are specimens of logarithms. For each state the characteristic and mantissa. Verify the correctness of the characteristics by the table in Fig. 26.

1. log 6315 = 3.8004.

2. log 65.24 = 1.8145.

3. log 6.608 = 0.8201.

4. log 2.36 = 0.3729.

5. log 48.731 = 1.6878.

6. log 1340.3 = 3.1272.

How to Find the Logarithm of a Number

82. A rule for determining the characteristic. The following is a table of numbers and characteristics of their logarithms:

Numbers..................	.0001 . . .	.001 . . .	.01 . . .	.1 . .	1 . . .
Characteristics............	−4	−3	−2	−1	0
Numbers..................	10 . . .	100 . . .	1000 . . .	10,000 . . .	100,000 . . .
Characteristics............	1	2	3	4	5

The table states that the logarithms of all numbers from 10 to 100, excluding 100, have the characteristic 1, of all numbers from 100 to 1000, excluding 1000, have the characteristic 2, etc. You will notice that in each case you can find the characteristic by counting: Start from the unit place and count 0, 1, 2, etc., until you come to the 1. If to reach the 1 you must count to the *left*, the characteristic is *positive*, if to the *right*, it is *negative*. The arrow in the table below shows the direction of counting.

Numbers	.0001	.001	.01	1	1
Counting	→ 0, 1, 2, 3, 4	→ 0, 1, 2, 3	→ 0, 1, 2	→ 0, 1	0
Characteristics	−4	−3	−2	−1	0
Numbers	10	100	1000	10,000	100,000
Counting	← 1, 0	← 2, 1, 0	← 3, 2, 1, 0	← 4, 3, 2, 1, 0	← 5, 4, 3, 2, 1, 0
Characteristics	1	2	3	4	5

Note that in each case the 1 is the *left-most figure* which is not zero.

Similarly, determine the characteristics of the numbers below, *i.e.*, start from the unit place and count until you reach the left-most figure which is not a zero.

Numbers	428. 3	0.684	0.0453	3678.4	4.315
Counting	← 2, 1, 0	→ 0, 1	→ 0, 1, 2	← 3, 2, 1, 0	0
Characteristics	2	−1	−2	3	0

The rule for determining the characteristic of a logarithm may now be stated in the final form: *To find the arithmetical value of the characteristic start at the unit*

place of the number and count 0, 1, 2, etc., until the left-most figure which is not a zero is reached. The sign of the characteristic is + or − according as you count to the left or right.

The preceding discussion makes it clear that the value of the characteristic depends entirely on the *position of the decimal point* in the number.

Exercises

Find the characteristics of the logarithms of the following numbers:

1. 6531.8.	**5.** 0.859.	**9.** 0.00952.
2. 3485.	**6.** 0.0642.	**10.** 0.000463.
3. 3072.	**7.** 1.648.	**11.** 0.0406.
4. 429.634.	**8.** 42.241.	**12.** 0.003005.

83. The mantissa does not depend upon the position of the decimal point. You have seen that the characteristic of a logarithm indicates the position of the decimal point in the number. The following shows that the mantissa is independent of the decimal point.

It is known that

$$\log 4326 = 3.6361, \quad i.e., \quad 10^{3.6361} = 4326.$$

Dividing the last equation by 10, you have

$$\frac{10^{3.6361}}{10^1} = \frac{4326}{10},$$

or $$10^{2.6361} = 432.6, \ i.e., \ \log 432.6 = 2.6361.$$

Dividing again by 10, you have

$$10^{1.6361} = 43.26, \ i.e., \ \log 43.26 = 1.6361.$$

Likewise, $$10^{.6361} = 4.326, \ i.e., \ \log 4.326 = .6361.$$

Thus, whenever you divide by 10 or by a power of 10, you change the position of the decimal point in the number and the characteristic of the logarithm, but leave the mantissa the same. The numbers 432.6, 43.26, 4.326, etc., all have the *same* mantissa. In general, it is said that *numbers which differ only in the position of the decimal point have the same mantissa.*

Mantissas are computed by methods studied in advanced courses in mathematics. We are interested only in the use of the mantissas in computations. For this purpose they have been arranged in tabular form on pp. 126 and 127.

84. The table of logarithms. The four-figure numbers in the tables (pp. 126 and 127) are mantissas approximated to four places of decimals. For convenience the decimal point has been left off. It should be understood. Thus, the mantissa corresponding to 35 is .5441. For ordinary purposes it is not necessary to have the mantissas carried beyond the fourth decimal place. When greater accuracy is required, larger tables must be used in which the mantissas are worked out to five or more places.

85. To find the logarithm of a two-figure number. To find the logarithm of 42, determine first the characteristic by rule. This gives 1. Then locate 42 in the column headed N and to the right, in the column headed 0, find the mantissa .6232.

$$\therefore \log 42 = 1.6232.$$

Exercises

Verify the following:

1. $\log 4.2 = 0.6232$.
2. $\log 89 = 1.9494$.
3. $\log 6.3 = 0.7993$.
4. $\log 63 = 1.7993$.
5. $\log 56 = 1.7482$.
6. $\log 7.2 = 0.8513$.
7. $\log 10 = 1.0000$.
8. $\log 200 = 2.3010$.
9. $\log 630 = 2.7993$.
10. $\log 100 = 2.0000$.

86. To find the logarithm of a three-figure number. The first two figures of the numbers whose logarithms are to be found are in column N. The third figures are in the top row of the table. The column headed 0 contains the mantissas for the numbers 0 to 100. The mantissas for the numbers 100 to 109 are in the first row, for 110 to 119 in the second row, for 120 to 129 in the third, etc. To find the mantissa of 438 locate 43 in column N, pass to the right to column headed 8, where you find 6415.

$$\therefore \log 438 = 2.6415.$$

Exercises

Verify the following:

1. $\log 34.6 = 1.5391$.
2. $\log 98.3 = 1.9926$.
3. $\log 532 = 2.7259$.
4. $\log 5.87 = 0.7686$.
5. $\log 3.65 = 0.5623$.
6. $\log 284 = 2.4533$.
7. $\log 340 = 2.5315$.
8. $\log 2.50 = 0.3979$.

87. To find the logarithm of a four-figure number. The examples which are worked out in full on the following page show how to find the logarithm of a number containing four figures.

1. Find log 338.2.

Solution: The mantissa of 3382 lies $\frac{2}{10}$ of the way between the mantissas of 3380 and 3390. In the table you find the mantissa of $3380 = 5289$ and the mantissa of $3390 = 5302$.

$\therefore$ the mantissa of $3382 = 5289 + \frac{2}{10}(5302 - 5289)$

$$= 5289 + \frac{2}{10}(13) = 5289 + 2.6.$$

Since 2.6 is nearer to 3 than to 2, add 5289 and 3.

$\therefore \log\ 338.2 = 2.5292.$

Nearly all of the work of finding a logarithm should be done mentally. In practice you should write only the most necessary parts as shown below.

Solution: *Write* log 338.2 =

The characteristic is 2. *Write* 2 to the right of the equality sign.

In row 33, column headed 8, find the mantissa 5289.

The next mantissa in the table is 5302.

The tabular difference is 13.

$\frac{2}{10}$ of $13 = 2.6$.

Drop the 6 and increase the 2 to 3. Add 3 to 5289.

Write 5292 to the right of the characteristic.

2. Find log 46.18.

Solution: Write log 46.18 =

The characteristic is 1. Write the 1.

In row 46, column headed 1, find 6637.

Next to it find 6646.

The tabular difference is 9.

$\frac{8}{10}$ of $9 = 7.2$.

Drop the 2, add 7 to 6637, and write 6644.

$\therefore \log 46.18 = 1.6644.$

LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474

LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0000	0004	0008	0013	0017	0021	0026	0030	0035	0039

Exercises

Find the following logarithms:

1. log 5.463.
2. log 3628.
3. log 80.31.
4. log 933.0.
5. log 18.42.
6. log 289.4.
7. log 3.521.
8. log 40.54.
9. log 236.5.
10. log 2995.
11. log 5.613.
12. log 629.1.
13. log 3.062.
14. log 8774.
15. log 20.49.
16. log 4.206.
17. log 135.2.
18. log 98.15.
19. log 6487.
20. log 2.165.
21. log 32.93.
22. log 7.805.
23. log 764.2.
24. log 457.0.
25. log 80.28.
26. log 8.758.
27. log 431.9.

88. Logarithms with negative characteristics. When the characteristic of a logarithm is negative it simplifies the work of computation to let $-1=9-10$, $-2=8-10$, $-3=7-10$, $-4=6-10$, etc.

Thus, log .432 has the characteristic -1 and the mantissa .6355.

$$\therefore \log .432=-1+.6355=9.6355-10.$$

Similarly, $\log .0432=8.6355-10$

and $\log .00432=7.6355-10.$

Exercises

Find the following logarithms:

1. log .0541.

Solution: The characteristic is -2, or $8-10$.
The mantissa is .7332.
$\therefore \log .0541=8.7332-10.$

2. log .362.	**8.** log .0006958	**14.** log .03416.
3. log .0587.	**9.** log 9.582.	**15.** log 75.34.
4. log .00364.	**10.** log .006524.	**16.** log .8002.
5. log .04582.	**11.** log 8.317.	**17.** log .3200.
6. log .6329.	**12.** log .9423.	**18.** log 28.00.
7. log .2583.	**13.** log 42.28.	**19.** log .07901.

89. How to find the number corresponding to a given logarithm. This is the reverse of the process of finding a logarithm. The following examples show the method:

1. Find the number whose logarithm is 2.6599.

Solution: Look among the mantissas until you find 6599.

Pass to the left and, in column N, find 45. Hence, the first 2 figures of the required number are 4, 5.

Pass upward from 6599 and find 7 at the top of the column. This is the third figure of the required number.

Hence, 4, 5, 7 are the figures of the required number.

The characteristic is now used to place the decimal point: Starting from the left-most figure, 4, count to the right: 0, 1, 2, and then place the decimal point to the right.

∴ The required number is 457.

The solution may now be repeated by omitting all explanations:

Among the mantissas find 6599.

In column N to the left find 45. *Write:* 45.

At the top find 7. *Write:* 7.

Beginning at 4 count to the right: 0, 1, 2, and place the decimal point to the right of the 7.

2. Find the number whose logarithm is 4.5092.

Solution: Among the mantissas find 5092.

To the left, in column N, find 32. *Write:* 32.

At the top find 3. *Write:* 3.

Beginning at the left-most figure count to the right: 0, 1, 2, 3, 4, add two zeros, and place the decimal point to the right.

The required number is 32300.

Similarly, find the numbers whose logarithms are 1.6107; 5.5465; 3.7143.

3. Find the number whose logarithm is $8.6637-10$.

Solution: Find the mantissa 6637 in the table.

To the left find 48; at the top 1.

Hence, the first three figures of the required number are 4, 8, 1.

Since the characteristic is -2 start, as before, from the left-most figure and count to the *left:* 0, 1, 2. In doing this place two zeros to the left of 481. This gives the figures 0, 0, 4, 8, 1. Then place the decimal point to the right, obtaining the required number: 0.0481.

Note that the essential difference between this problem and the preceding is the *direction* of counting.

Verify the following:

$8.5658-10=\log 0.0368$.

$9.7372-10=\log 0.546$.

$7.3483-10=\log 0.00223$.

4. Find the number whose logarithm is 0.3201.

Solution: Locate the mantissa 3201 in the table. *Write:* 209.

Beginning with the left-most figure, 2, count 0 and place the decimal point to the right.

Hence, $0.3201=\log 2.09$.

Find the numbers corresponding to the following logarithms: 0.4900; 0.5502; 0.4014.

JOHN NAPIER

John Napier (1550–1617), a wealthy Scotch baron, was the inventor of logarithms. He had tried for years to find simple ways of performing multiplications and divisions. He derived his logarithms from the study of arithmetical and geometric progressions and the binomial theorem, not from exponents; for the theory of exponents was not sufficiently developed and understood in his time. The table of common logarithms with the base 10 was constructed by Henry Briggs (1560–1630), an English professor, who had read Napier's work and had gone to Scotland for a conference with Napier about his discovery. But the whole world is indebted to Napier for his great invention.

5. Find the number whose logarithm is 2.4238.

Solution: Locate the mantissa which is less than 4238, but nearest to it. This is 4232.

To the left find 26; on top find 5. *Write:* 265.

Hence, the required number lies between 2650 and 2660.

The difference between the mantissas is $4238-4232=6$.

The tabular difference is $4249-4238=11$.

$$\therefore \frac{6}{11}=\frac{x}{10},$$

where x is the fourth figure of the required number.

$\therefore x=\frac{60}{11}=5.5$, or 5 approximately.

The required number is made up of the figures 2, 6, 5, 5.

Placing the decimal point, we find it to be 265.5.

Exercises

Find the numbers corresponding to the following logarithms:

1. 0.6341.	**6.** 2.3729.	**11.** 8.8077 − 10.
2. 1.8096.	**7.** 1.9557.	**12.** 1.4900.
3. 2.8865.	**8.** 0.5240.	**13.** 9.5717 − 10.
4. 8.1761 − 10.	**9.** 9.7973 − 10.	**14.** 8.3090 − 10.
5. 0.6970.	**10.** 2.8192.	**15.** 2.4527.

Computation by Means of Common Logarithms

90. How to multiply numbers by means of logarithms. You have seen (§76) that numbers can be multiplied by means of a table of exponents, or logarithms. The following steps are involved:

1. The logarithms are found and added.
2. The number corresponding to the sum is found. This is the required product.

Thus, the work of finding the product $42.3 \times 6.89 \times .341$ may be arranged as follows:

Outline:

$$\begin{aligned} \log 42.3 &= \\ \log 6.89 &= \\ \log .341 &= \\ \hline \text{The sum} &= \\ \text{The required product} &= \end{aligned}$$

In all computation the outline should be made before the table is used.

Verify the following solution:

Solution:

$$\begin{aligned} \log 42.3 &= 1.6263 \\ \log 6.89 &= 0.8382 \\ \log .341 &= 9.5328 - 10 \\ \hline \log N &= 11.9973 - 10 \\ &= 1.9973. \qquad \therefore N = 99.38. \end{aligned}$$

Exercises

Find the following products:

1. $13.16 \times 8.742 \times .0153$.
2. $1.38 \times 84.32 \times .651$.
3. $25.1 \times 16.38 \times 2.416$.
4. $.6462 \times .9865 \times 12.22$.
5. $1.89 \times 3.724 \times 4.987$.
6. $8.675 \times 53.67 \times .1832$.
7. $15.8 \times 2.53 \times 12.84$.
8. $64.91 \times 3.82 \times 1.74$.
9. $1.582 \times 23.84 \times 5.46$.
10. $.00832 \times .0469 \times 128.2$.

91. How to divide numbers by means of logarithms. You have seen (§77) that exponents, or logarithms, enable you to change the process of dividing numbers to that of subtracting. The following steps are involved:

1. Find the logarithms of dividend and divisor.
2. Subtract the second from the first.
3. Find the number corresponding to the difference.

Thus the work of dividing 3.18 by 1.63 may be arranged as below:

Outline:
$$\begin{array}{r l} \log 3.18 = & \\ \log 1.63 = & \\ \hline \text{Difference} = & \\ \text{Quotient} = & \end{array}$$

Exercises

Find the following quotients:

1. $389.6 \div 4.265$.

Solution:
$$\begin{array}{r} \log 389.6 = 2.5906 \\ \log 4.265 = 0.6299 \\ \hline \log Q = 1.9607 \\ Q = 91.35. \end{array}$$

2. $398.6 \div 4.533$. **5.** $16.51 \div 2.186$. **8.** $2.958 \div 1.736$.

3. $54.82 \div 26.01$. **6.** $22.22 \div 8.301$. **9.** $5864 \div 3421$.

4. $3.852 \div 3.735$. **7.** $256.2 \div 144.8$. **10.** $900.6 \div 31.24$.

11. $8.431 \div .6324$.

In the solution below the characteristic of log 8.431 is changed from 0 to $10-10$ to simplify the subtraction.

Solution:
$$\begin{array}{rl} \log 8.431 = & 10.9259-10 \\ \log .6324 = & 9.8010-10 \\ \hline \log Q = & 1.1249 \\ Q = & 13.33. \end{array}$$

12. $42.64 \div 85.63$.

Solution:
$$\begin{array}{rl} \log 42.64 = & 11.6298-10 \\ \log 85.63 = & 1.9326 \\ \hline \log Q = & 9.6972-10 \\ Q = & .4980. \end{array}$$

13. $40.31 \div .3172$. **15.** $368.8 \div 462.8$. **17.** $.00358 \div .000878$.

14. $.0649 \div .485$. **16.** $.08321 \div .0942$. **18.** $.6427 \div .7831$.

Perform the following operations by means of logarithms:

19. $F = \dfrac{3.48 \times 1.97 \times 6.314}{.8591 \times .324 \times 41.65}$.

Suggestion: Make an outline as below. Then use the table.

Outline:
$$\begin{array}{rl} \log 3.48 = & \\ \log 1.97 = & \\ \log 6.314 = & \\ \hline \log \text{numerator} = & \\ \log \text{denominator} = & \\ \hline \log F = & \\ F = & \end{array} \qquad \begin{array}{rl} \log .8591 = & \\ \log .324 = & \\ \log 41.65 = & \\ \hline \log \text{denominator} = & \end{array}$$

20. $\dfrac{36.42\times 28.4}{32.5\times 31.4}$.

21. $\dfrac{364.1\times 4.87}{68.1\times 32.58}$.

22. $\dfrac{16.5\times .832\times .0531}{281\times .0466}$.

23. $\dfrac{1.306\times 208\times .00581}{3014\times .0592\times 3.61}$.

24. $\dfrac{3.145\times 8\times 32.4}{1.042\times 23\times 16.1}$.

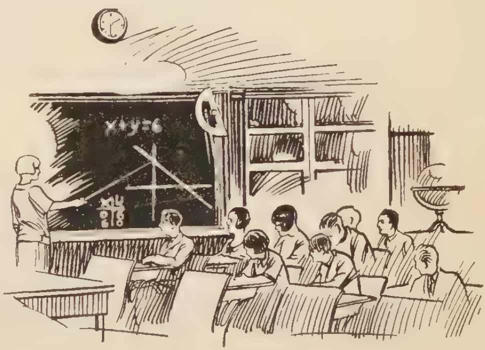
25. $\dfrac{.00643\times .0124\times .00048}{3.612\times .000312\times .000894}$.

Solve the following problems by logarithms:

26. The area of a triangle is given by the formula $A=\dfrac{bh}{2}$. Find A if $b=442.3$ feet and $h=361.8$ feet.

27. The perimeter of a triangle is 438.1 units and the radius of the inscribed circle is 39.05 units. Using the formula $A=rs$, where s denotes the semiperimeter, find A.

28. The total area of a cylinder is $T=2\pi r(r+h)$. Find the total area of a cylinder whose height is 981 centimeters and whose base radius is 490 centimeters. Use $\pi=3.14$.



29. Find the number of cubic feet of air in a classroom whose dimensions are 48 feet by 25.5 feet by 16.5 feet.

30. Find the simple interest on \$865, at 5%, for $6\frac{1}{4}$ years.

92. How to find a power of a number. To illustrate the method, let us find the value of 3^4.

Since $3^4=3\times 3\times 3\times 3$, the logarithm of 3^4 is $\log 3+\log 3+\log 3+\log 3$, or $4\log 3$, *i.e., the logarithm*

of a power is found by multiplying the exponent by the logarithm of the base.

The following is another way of seeing the truth of this principle:

Let $m = \log 3$.

This means that

$$10^m = 3.$$

$\therefore 10^{4m} = 3^4$ (by raising both members to the fourth power).

The last equation means that $\log 3^4 = 4m = 4 \log 3$, which is the result obtained above.

To find the value of 3^4 proceed as follows:

Solution: $\log 3 = 0.4771.$

$\log 3^4$, or $4 \log 3 = 1.9084.$

The number corresponding to this logarithm is $3^4 = 81.0.$

Exercises

Find the following powers by means of logarithms:

1. 6^3.	4. 29^2.	7. 9^3.
2. 18^2.	5. 7^3.	8. 3^6.
3. 23^2.	6. 4^4.	9. 6^4.

93. Changing the root of a number into a power. If we assume that the law $a^m \cdot a^n = a^{m+n}$ holds for fractional exponents, then

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a,$$
$$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a,$$
$$a^{\frac{1}{4}} \cdot a^{\frac{1}{4}} \cdot a^{\frac{1}{4}} \cdot a^{\frac{1}{4}} = a.$$

Thus, $a^{\frac{1}{2}}$ is one of the 2 equal factors whose product is a,

$a^{\frac{1}{3}}$ is one of the 3 equal factors whose product is a,

$a^{\frac{1}{4}}$ is one of the 4 equal factors whose product is a.

This means that $a^{\frac{1}{2}}$ is only another symbol for the square root of a, *i.e.*, $a^{\frac{1}{2}}=\sqrt{a}$.

Similarly, $a^{\frac{1}{3}}$ is the same as $\sqrt[3]{a}$; $a^{\frac{1}{4}}$ is the same as $\sqrt[4]{a}$.

In general, $a^{\frac{1}{n}}=\sqrt[n]{a}$.

94. How to use logarithms to find a root of a number. Since a root can be changed to a power with a fractional exponent, you can use the method of §92.

The following example illustrates the process:

To find $\sqrt{8281}$.

Outline: Change $\sqrt{8281}$ to the form $8281^{\frac{1}{2}}$.

Find log 8281 =

Then find $\frac{1}{2}$(log 8281) =

Finally, find the number corresponding to this logarithm.

In computation the first two steps are to be taken mentally. This leaves the solution in the form shown below:

Solution: log 8281 = 3.9181.

$\frac{1}{2}$(log 8281) = 1.9590.

$\therefore \sqrt{8281} = 91.00$.

Exercises

Find the value of each of the following:

1. $\sqrt[5]{328.4}$.

Suggestion: log 328.4 =
$\frac{1}{5}$(log 328.4) =
$\sqrt[5]{328.4}$ =

2. $\sqrt[3]{4.325}$. 3. $\sqrt[4]{9.863}$. 4. $\sqrt{9584}$. 5. $\sqrt[4]{8.762}$.

6. $\sqrt{.0786}$.

Suggestion: log .0786 = 8.8949−10.
Change to 18.8949−20.
Dividing by 2, you have 9.4474−10.

7. $\sqrt{.00429}$. 8. $\sqrt{.643}$. 9. $\sqrt{.00582}$.

10. $\sqrt{3\times 4.12\times 7.14}$.

Suggestion: log 3 =
log 4.12 =
log 7.14 =

Sum =
$\frac{1}{2}$(Sum) =
$\sqrt{3\times 4.12\times 7.14}$ =

11. $\sqrt[3]{6.83\times 294.3}$. 12. $\sqrt[4]{64.8\times 7.34}$. 13. $\sqrt{841\times 36\times 15}$.

14. $\sqrt{\dfrac{283\times 4.631}{7.61^3}}$.

Suggestion: log 283 =
log 4.631 =

Sum =
3(log 7.61) =

Difference =
$\frac{1}{2}$(Difference) =
$\sqrt{\dfrac{283\times 4.631}{7.61^3}}$ =

log 7.61 =
3(log 7.61) =

15. $\sqrt{\dfrac{576\times 8.14^2}{2.64\times 13.7^2}}$.

16. $\sqrt[3]{\dfrac{36.75\times 25.6}{27.2\times 18.67}}$.

17. $\sqrt{\dfrac{3.14\times 27.31}{2.738\times 1.042\times 12.4}}$.

18. $\sqrt[3]{\dfrac{.0581\times .00356}{.987\times .00421}}$.

95. The use of logarithms in problems. In the following problems logarithms should be used in all operations except in additions and subtractions.

Exercises

1. The sides of a triangle are 82.4 feet, 63.1 feet, and 78.1 feet. Find the area by means of the formula $\sqrt{s(s-a)(s-b)(s-c)}$.

2. The volume of a sphere is $V=\frac{4}{3}\pi r^3$. Find the volume of a metal sphere whose radius is 3.26 inches. Use $\pi=3.14$.

3. One of the sides of a right triangle is found by the formula $a=\sqrt{c^2-b^2}=\sqrt{(c+b)(c-d)}$. Find the side a if the hypotenuse is 892.3 yards and if the other side is 627.4 yards.

4. The amount A of an investment p, after n years, with an annual interest rate r, compounded k times a year, is given by the formula $A=p\left(1+\frac{r}{k}\right)^{kn}$. Find the amount after 18 years, on an investment of \$3760, with interest at 5%, compounded quarterly.

Suggestion: $p=3760$, $r=\frac{5}{100}$, $k=4$, $n=18$.

5. Find the amount after 10 years, on an investment of \$600, with interest at 5% compounded annually.

6. What sum must be invested to yield \$5500, in 25 years, at an interest rate of 4%, compounded annually?

Suggestion: Solve the equation in Exercise 4 for p. Then use logarithms.

7. What sum would yield \$20,000 in 30 years, at 5% interest, compounded semi-annually?

8. Find the sum which must be set aside for the education of a boy at the age of 1 year to yield \$2500, in 16 years, at 4% interest, compounded quarterly.

9. At what rate of interest, compounded semi-annually, would an investment of \$2250 yield \$5500 after 20 years?

Solution: Using the formula of Exercise 4, you have

$$5500 = 2250\left(1+\frac{r}{2}\right)^{40}$$

$$\therefore \frac{\overset{22}{\cancel{\overset{110}{\cancel{5500}}}}}{\underset{9}{\cancel{\underset{45}{\cancel{2250}}}}} = \left(1+\frac{r}{2}\right)^{40}$$

$$\therefore \left(1+\frac{r}{2}\right)^{40} = \frac{22}{9}$$

$$1+\frac{r}{2} = \sqrt[40]{\frac{22}{9}}$$

Use logarithms to find $\sqrt[40]{\frac{22}{9}}$. Then find r.

10. At what rate will \$95, compounded annually, amount to \$135 in 8 years?

11. The formula $d = \sqrt[3]{\frac{65h}{n}}$ gives the number of inches in the diameter of a shaft required to transmit h horse power at a speed of n revolutions per minute. What diameter is required to transmit 120 horse power at a rate of 115 revolutions per minute?

12. The approximate velocity v of an object which has fallen s feet is found from the formula $v = \sqrt{2gs}$. Find the velocity of an object which has fallen 350 feet. Let $g = 32.16$.

96. What every pupil should be able to do. After completing the study of Chapter VI you should be able to do the following:

1. To find the logarithm of a given number.

2. To find the number corresponding to a given logarithm.

3. To use logarithms to perform the processes of multiplying, dividing, raising to a power, and extracting roots.

4. To solve problems involving formulas which are more easily evaluated by logarithms than by actually performing the indicated operations.

97. Typical problems and exercises. You should be able to solve with ease exercises of the type given below:

1. Find the logarithms of 73.52; .7352; .007352.

2. Find the numbers whose logarithms are 2.4849; $9.4849-10$; $7.4849-10$.

3. Find the product $12.24 \times 0.0583 \times 28.12$.

4. Find the quotient $9.643 \div .8416$.

Find the value of the following:

5. $\dfrac{26.31 \times 1.289}{78.42 \times .3651}$.

6. $68^2 \times \sqrt[3]{249}$.

7. The volume of a cylinder is given by the formula $v = \pi r^2 h$. Find the volume if $r = 1.36$ inches and $h = 16$ inches.

8. Simple interest is computed by means of the formula $i = prt$. Find i when $p = 5283$, $r = .06$, $t = 8$.

9. The lateral area of a frustum of a cone is given by the formula $L = \pi s\left(\dfrac{D+d}{2}\right)$. Find the lateral area if the diameters of the bases are $D = 6.21$ inches, $d = 5.38$ inches, the slant height $s = 5.38$ inches. Take $\pi = 3.14$.

CHAPTER VII

THE SLIDE RULE*

The Logarithmic Scale

98. Where the slide rule is used. In Chapter VI you have learned to perform by means of logarithms the operations of multiplying, dividing, extracting roots, and finding powers. People whose work calls for a large amount of computation use freely any short cuts and devices that save time and labor.

Various mechanical computing machines have been invented to do much of the arithmetical work for them. One machine familiar to all of us is the adding machine used in stores and offices. Other devices for making computations are tables of powers, roots, and logarithms.

One of the best instruments for performing difficult arithmetical processes with ease and speed is the *slide rule* (Fig. 32).† It is being used more and more by

*If a class studies Chapter VII, each pupil should have a slide rule. The Keuffel and Esser Co., New York, or Chicago, sells a good student slide rule at a moderate price.

†William Oughtred (1574–1660) is the inventor of the slide rule. He was one of the greatest writers of mathematics in the early part of the 16th century. He wrote books on arithmetic, algebra, and logarithms. Although he studied much and slept little, he reached the old age of eighty-six years.

statisticians and others engaged in scientific work, as well as by men doing practical work in shops and offices. In school work you will find the slide rule helpful wherever arithmetical computations arise.

99. How to construct a logarithmic scale. The logarithmic scale is really a table of logarithms marked off to scale along the edge of a ruler. The following table gives the logarithms of the whole numbers 1, 2, 3, . . . 10:

Numbers. . .	1	2	3	4	5	6	7	8	9	10
Logarithms .	0	0.3010	0.4771	0.6021	0.6990	0.7782	0.8451	0.9031	0.9542	1

Let the length of AB (Fig. 27) represent the unit 1. Then $AB = 1 = \log 10$. It is immaterial what length is selected for AB.

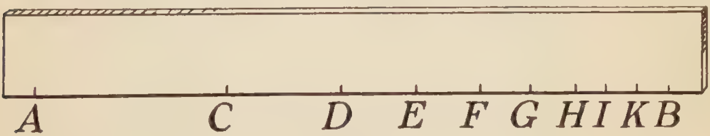


FIG. 27

Similarly,

$AC = .30 = \log 2$ $\quad$ $AG = .78 = \log 6$

$AD = .48 = \log 3$ $\quad$ $AH = .85 = \log 7$

$AE = .60 = \log 4$ $\quad$ $AI = .90 = \log 8$

$AF = .70 = \log 5$ $\quad$ $AK = .95 = \log 9$

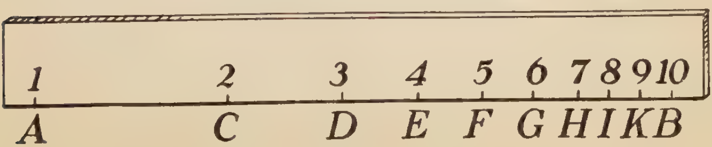


FIG. 28

In Fig. 28 the numbers whose logarithms are represented by AC, AD, etc., are written above the points C, D, etc. Since log 1 is 0, the number 1 is written over point A.

Thus, the distance from the index 1 to any number is the logarithm of that number. AB is a *logarithmic scale.*

100. A close examination of a logarithmic scale. The divisions marked off on the logarithmic scale (Fig. 28) may be subdivided as shown in Fig. 29. To place the logarithms of 1.1, 1.2, 1.3, . . . 1.9 on the scale mark off the numbers 0.0414, 0.0792, 0.1139, . . . 0.2788. Then $AM = 0.176 = \log 1.5$.

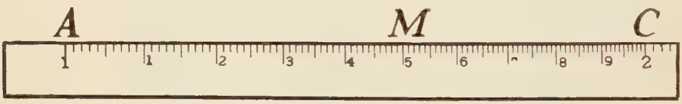


FIG. 29

Similarly, subdivide the distances CD, DE, . . . KB, and let the subdivisions represent the logarithms of 2.1, 2.2, 2.3 . . . ; 3.1, 3.2, 3.3, . . . You will then have on the logarithmic scale the logarithms of 2-figure numbers.

To represent the logarithms of three-figure numbers we must subdivide again. If you continue to carry this to more than three figures, the divisions become too small to be of practical use. Furthermore, for most computations which have to be made with numbers found by measurement, the results are sufficiently accurate if given in three-figure numbers. A magnifying glass may be used to take the readings of the small divisions.

By making AB (Fig. 27) sufficiently large the graduations may be increased and a greater degree of accuracy attained in reading off the numbers corresponding to given logarithms. A scale 10 inches long gives the numbers to three figures. When greater accuracy is required a more elaborate instrument has to be used.

How the Slide Rule Is Constructed

101. A description of the slide rule. The slide rule (Fig. 32) has four logarithmic scales A, B, C, and D. Scales C and D are single scales (Fig. 30) extending from 1 to 10. Scales A and B are double (Fig. 31) extending from 1 to 100.

FIG. 30

FIG. 31

That the two parts of scales A or B must be identically the same follows from the fact that the number pairs 1, 10; 2, 20; 3, 30; etc., have the same mantissa but different characteristics.

Thus, from the table you find that

$$\begin{aligned} \log\ 2 &= 0.3031 \\ \log 10 &= 1.0000 \\ \log 20 &= 1.3031 \\ \therefore \log 20 &= \log 10 + \log 2. \end{aligned}$$

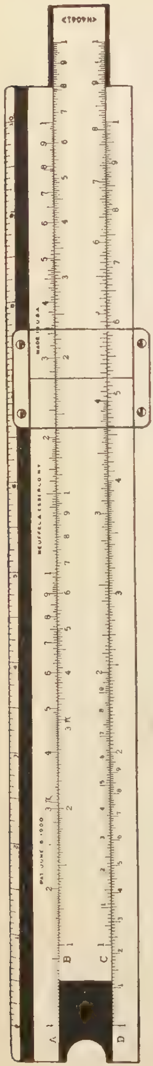


FIG. 32

Similarly, to locate log 30 on the scale we lay off log 10 first and then log 3 in the same direction. Thus, scale *A* should read: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100.

Note that a segment representing a logarithm on scale *C* is twice as large as the segment representing the same logarithm on scale *A*. Furthermore, log 6 on scale *C* is equal to 2 log 6, or log 6^2, or log 36, on scale *A*.

Scales *A* and *D* are fixed, but scales *B* and *C* can be made to slide.

The taking of the readings is facilitated by using the runner *R* (Fig. 30) which can be moved to any desired point on the scales.

EXERCISES

1. Locate the following two-figure numbers on scale *C*:

2	4	7	3.4	3.7	3.9
1.2	1.5	1.9	4.9	5.2	7.1
2.3	2.5	2.8	8.4	9.8	9.5

2. Locate on scale *A* the numbers given in Exercise 1.

3. Locate on scale *A* the following numbers:

12	15	19	49	52	61
23	25	28	78	82	87
34	37	39	91	95	99

4. Locate on scale C:

1.24	2.95	6.45
1.83	2.72	8.35
1.76	2.84	9.15

Performing Operations with the Slide Rule

102. How to multiply with a slide rule. It must be remembered that the scales on the slide rule are logarithmic tables represented geometrically. The numbers whose logarithms are required are printed on the scales. The mantissas are the lengths on the scale counted from the index 1 on the left end. The characteristics are determined by inspection.

Scales C and D are constructed to a larger scale than A and B. Therefore, more accurate results can be obtained with them in multiplying numbers than with A and B.

The method of multiplying is illustrated in the following examples:

1. Find the product 2×3.

Solution: Move the sliding scale C until the index 1 of scale C is exactly over 2 on scale D (Fig. 33).

Fig. 33

Pass along scale C to 3. Directly under 3 is the number 6. This is the required product.

Explanation: The distance from 1 to 2 on scale *D* is log 2.

The distance from 1 to 3 on scale *C* is equal to log 3.

Since the sum of the two distances is equal to the distance from 1 to 6 on scale *D*, it follows that this is equal to log 2+log 3 or log 2×3, or log 6.

2. Find the product 1.65×2.34.

Solution: Place the runner on 1.65, scale *D* (Fig. 34).

Move scale *C* until the index 1 is exactly above 1.65 on scale *D*.

Move the runner to 2.34, scale *C*.

Then take the reading shown by the runner on scale *D*.

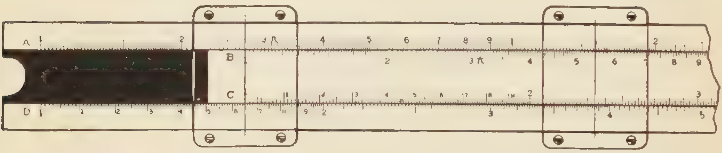


FIG. 34

The result is 3.86 approximately. Verify the result by actual multiplication.

3. Find the product 2.4×3×1.61.

Solution: Place the runner to 2.4 on scale *D*.

Move scale *C* until the index 1 is exactly over 2.4.

Move the runner to 3 on scale *C*.

Move scale *C* until the index 1 is under the line on the runner.

Move the runner to 1.61 on scale *C*.

The runner now extends beyond scale *D* and the product 2.4×3×1.61 cannot be read off. Whenever this happens move the runner back to the preceding

position, in this case to 2.4×3 on scale D. Then move the slide back until the index 1 on the *right* is under the line on the runner. Then move the runner to 1.61 on scale C.

On scale D the reading indicated by the runner shows the first three figures of the product to be 116.

Place the decimal point by inspection, and obtain the final result 11.6.

Exercises

Find the following products with the slide rule:

1.	1.62×2.48	**8.**	2.6×9.2	**15.**	15.1×12.6
2.	$.45 \times 1.37$	**9.**	5.4×3.12	**16.**	$4 \times 2 \times 3$
3.	4.16×2.13	**10.**	1.75×8.6	**17.**	$5.1 \times 4 \times 1.2$
4.	6.15×1.32	**11.**	2.34×8.12	**18.**	$3.14 \times 6 \times 2.16$
5.	5.81×1.48	**12.**	6.41×2.15	**19.**	$4.8 \times 3.6 \times 2.5$
6.	6×8	**13.**	11.2×1.46	**20.**	$7.2 \times 4.1 \times 1.8$
7.	4×84	**14.**	8.36×4.14	**21.**	$3.6 \times 1.2 \times .84$

103. How to divide with the slide rule. Division is the inverse of multiplication. You have found previously (§77) that in performing a division by logarithms you proceed as in multiplication, but you *subtract* the logarithms instead of adding them. The use of the slide rule for performing a division will be shown in the following examples:

1. Find the quotient $8 \div 4$.

Solution: Locate 8 on scale D (Fig. 35).
Place the runner over 8.
Move the slide until the 4 is exactly above the 8.

Pass along scale C to the index 1.

Read on scale D the number below the index 1 on scale C.

This is the required quotient.

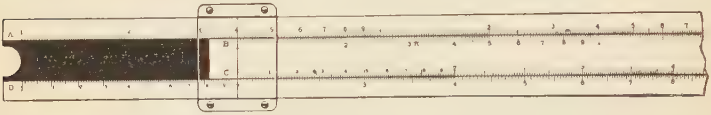


FIG. 35

2. Find the quotient $268 \div 24$.

Solution: As before locate 268 on scale D.

Move the slide until 24 falls above 268.

On scale D read the number exactly below the index 1 on scale C. This gives 112.

The decimal point is placed by inspection: The quotient $\frac{268}{24}$ must be approximately 10. Hence, the correct answer is 11.2.

EXERCISES

Find the following quotients with the slide rule:

1. $6 \div 3$	**5.** $10 \div 2.5$	**9.** $12.4 \div 8$	**13.** $14.1 \div 2.16$
2. $3 \div 6$	**6.** $16 \div 3.6$	**10.** $18.3 \div 5$	**14.** $7.18 \div 12.2$
3. $14 \div 7$	**7.** $14 \div 8.4$	**11.** $14.8 \div 3$	**15.** $16.5 \div 8.8$
4. $36 \div 9$	**8.** $18 \div 6.7$	**12.** $7.12 \div 6$	**16.** $9.42 \div 3.18$

104. How to solve proportions by means of the slide rule. Solving the proportion $\frac{a}{b} = \frac{c}{d}$ for one of the literal numbers, as a, you have $a = \frac{bc}{d}$. To find the value of $\frac{bc}{d}$, divide d into b and multiply the quotient by c.

The following example illustrates the method when the slide rule is used:

Find the value of x satisfying the equation

$$\frac{x}{3.42}=\frac{19.1}{61.6}$$

Solution: Solving for x, you have $x=\dfrac{19.1\times 3.42}{61.6}$.

Paying no attention to the decimal points in the numbers, move the runner to 191 on scale D (Fig. 36).

Move 616 on scale C to the runner.

Since you are not interested in the quotient, do not stop to read it but note the fact that it is on scale D below the index 1 of scale C.

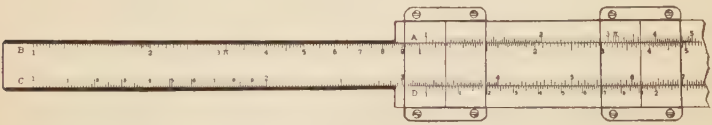


FIG. 36

Move the runner to 342 on scale C and take the reading on scale D. This is 106.

Locate the decimal point by inspection as follows: $19\div 61$ is about $\frac{1}{3}$, and $\frac{1}{3}\times 3$ is 1.

Hence, the result is 1.06.

EXERCISES

Find the value of x in each of the following:

1. $x=.04\times 2000\times 4.5$.
2. $x=\dfrac{26.3\times 4.16}{98.1}$.
3. $x=\dfrac{42.6\times 3.41}{162}$.
4. $56.1x=18.6\times 3.42$.
5. $\dfrac{x}{18.2}=\dfrac{2.41}{36.5}$.
6. $\dfrac{x}{1.28}=\dfrac{21.5}{84.2}$.

105. How to carry on a continuous series of computations. The following example shows how a series of multiplications and divisions may be performed with the slide rule:

Find the value of

$$\frac{3.64 \times 14.9 \times 58 \times 0.314}{61.8 \times 0.038 \times 4.16}.$$

Paying no attention to the decimal points, move the runner to 364 on scale *D*.

Move scale *C* until 618 falls on the runner.

Place the runner at 149, scale *C*.

Move scale *C* until 38 falls on the runner.

Move runner to 58, scale *C*.

Move 416 on scale *C* to the runner.

Move runner to 314 on scale *C*.

On scale *D* take the reading indicated by the runner. This is 101.

Place the decimal point by inspection as follows:

$$3.64 \times 14.9 = 45 \text{ approximately,}$$
$$58 \times .314 = 20 \text{ approximately,}$$
$$45 \times 20 = 900.$$
$$61.8 \times 0.03 = 1.8 \text{ approximately,}$$
$$1.8 \times 4.16 = 8 \text{ approximately,}$$
$$900 \div 8 = 112 \text{ approximately.}$$

Hence, the answer is 101.

Exercises

Find the value of each of the following:

1. $\frac{3 \times 48}{7 \times 29}$ 2. $\frac{4 \times 26}{15 \times 3}$ 3. $\frac{6.1 \times 2.7}{1.25 \times 8.74}$

4. $\dfrac{10.2\times8.61}{21.4\times7.38}$ 6. $\dfrac{16.1\times0.032}{4.16\times3.84}$ 8. $\dfrac{16.5\times12.8\times0.34}{89.1\times2.11}$

5. $\dfrac{27.1\times15.3}{4.85\times81.2}$ 7. $\dfrac{72.4\times16.3\times4.12}{81.5\times50.7}$ 9. $\dfrac{0.46\times18.1\times4.15}{12.6\times3.18}$

106. How to find squares and square roots with a slide rule. Comparing scales *A* and *B* (Fig. 37) with *C* and *D* you notice that the numbers on the first two scales are the squares of those directly below on the

FIG. 37

last two. This is because the logarithms for scales *A* and *B* were laid off to a scale half as large as that used for *C* and *D*. Hence, to find the square of a number, locate the number on scale *D*, and move the runner to that point. Then read off the square on scale *A*.

Conversely, to find the square root of a number locate the number on scale *A* and with the help of the runner read the number directly below on scale *D*. This is the required square root. The following examples illustrate the process:

1. Find the square of 35.4.

Solution: Move the runner to 354 on scale *D*.
Take the reading directly above on scale *A*.
This is 125.
Hence, the answer is 1250 approximately.

2. Find the square root of 35.4.

Solution: Move the runner to 354 on scale A.

Directly below on scale D find 595.

The answer is 5.95.

3. Find the cube of 2.14.

Solution: Change $(2.14)^3$ to $(2.14)^2 \times 2.14$.

Move the runner to 2.14 on scale D.

Move the slide until the index 1 on scale B is on the runner.

Move the runner to 2.14 on scale B.

Take the reading directly above on scale A.

Exercises

Find the squares of the following numbers:

1. 3	4. 4.12	7. 0.34	10. 61.8
2. 5	5. 8.63	8. 0.21	11. 43.9
3. 4.1	6. 9.45	9. 0.56	12. 84.2

Find the square root of each of the following numbers:

13. 25	15. 3.16	17. 0.28	19. 35.2
14. 14	16. 4.28	18. 0.58	20. 61.8

107. Computations in problems performed with the slide rule. The computations in the problems below are readily carried out with the slide rule.

Exercises

Solve the following problems:

1. Find the diameter of a circle whose circumference is 34.3 inches.

Solution: $c = \pi d$.

$$\therefore d = \frac{c}{\pi} = \frac{34.3}{3.14}.$$

Use the slide rule to perform the division.

2. Find the area of a circle whose radius is 6.18.

3. A flagpole casts a shadow 28 feet long. At the same time a rod 5.6 feet high casts a shadow 4.3 feet long. Find the height of the pole.

4. Find the simple interest on \$328 at $4\frac{1}{2}\%$ for $3\frac{1}{2}$ years.

5. Find the radius of a circle whose area is 36.5 square feet.

6. What sum of money placed at simple interest at $4\frac{1}{2}\%$ yields an income of \$250 in 3 years?

7. A farm valued at \$12,300 is taxed for \$74.80. At the same rate what will be the tax on a farm valued at \$15,200?

8. If 225 pounds of milk produce 8.1 pounds of butter fat, how many pounds of milk will be required to produce 32 pounds of butter fat?

9. The pole of a tent is 32 feet high. To hold the pole in position stakes are placed 46 feet from the foot of the pole, and wires of equal length attached to the top of the pole are fastened to these stakes. Find the length of the wires.

10. Find the base of a triangle whose area is 30.5 square feet and whose altitude is 10.4 feet.

11. How many cubic feet of ground must be excavated to make a ditch 2.4 feet wide, 3.5 feet deep, and 124 feet long?

12. Find the cost of painting the lateral surface of a silo 14.5 feet in diameter, 35.8 feet high, at the rate of \$2.75 per 100 square feet.

13. Find the horse power of an eight cylinder engine, the diameter of whose pistons is 4.5 inches.

Suggestion: Use the formula $\text{h.p.} = \dfrac{d^2n}{2.5}\cdot$

14. Find the volume of a sphere whose diameter is 1.26 inches.

15. A screen is 16 feet from a lamp. How far from it must it be moved to receive twice as much light? See Exercise 3, §18.

16. Find the sum of 6 terms of an arithmetical progression whose first term is 3.14, the constant difference between consecutive terms being 2.25.

108. What every pupil should be able to do. The purpose of Chapter VII is to teach you to use the slide rule for such operations as are usually found in practical problems. It is expected that you should now be able to perform the following operations with the slide rule:

1. To find the product of two numbers.
2. To find the quotient of two numbers.
3. To find the square of a number.
4. To find the square root of a number.
5. To solve proportions.
6. To perform in succession the operations of a continuous series of multiplications and divisions.

CHAPTER VIII

PROBLEMS LEADING TO LINEAR EQUATIONS WITH SEVERAL UNKNOWNS

The Graphical Method of Solving Linear Equations

109. Solving two linear equations in two unknowns graphically. You have learned previously that the graph of an equation in two unknowns is a straight line, and that the solution of a pair of linear equations with two unknowns is represented by the point of intersection of two lines. It is clear that, if the graphs of two equations are two *parallel* lines, there can be no solution, since the lines have no point in common. The equations are then said to be *inconsistent.*

If the graphs of two equations are the *same* straight line, any solution of one equation is also a solution of the other, since the lines have all points in common. In that case the equations are *equivalent.* If you wish to learn more about inconsistent and equivalent equations read §§158 and 159.

If two equations are to be solved graphically the lines representing the equations must intersect. The following example explains the steps in the solution:

Solve the system of linear equations: $\begin{cases} 3x+7y=7 \\ 5x+3y=29. \end{cases}$

Solution: 1. Make a table of corresponding values of x and y (Fig. 38).

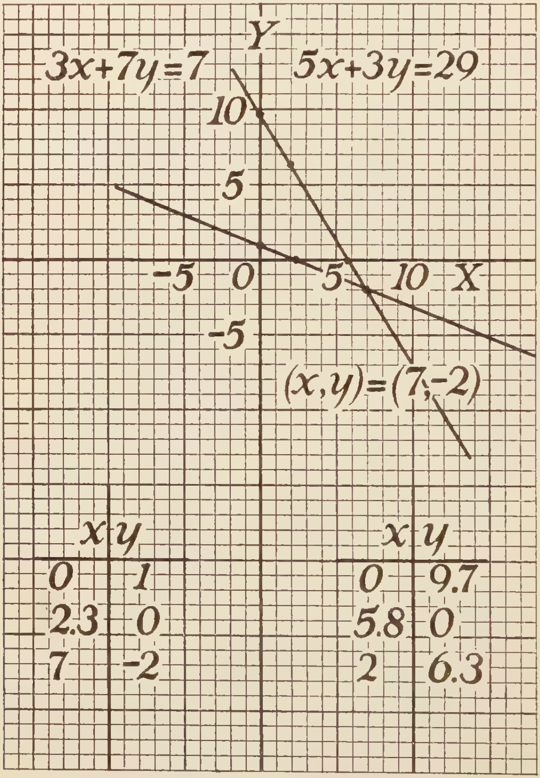


Fig. 38

Thus, in the first equation

let $x=0$. Then $y=1$.
Let $y=0$. Then $x=2.3$ approximately.
Let $x=7$. Then $y=-2$.

In the second equation

let $x=0$. Then $y=9.66$ approximately.
Let $y=0$. Then $x=5.8$.
Let $x=2$. Then $y=6.3$ approximately.

2. Draw the reference axes OX and OY, and lay off convenient units.

3. Plot the points representing the number pairs in the tables.

4. Draw the two lines passing through the plotted points, and locate the point of intersection.

5. Read off the values of x and y corresponding to the point of intersection. This gives the solution

$$(x, y)=(7, -2)$$

Exercises

Solve the following systems graphically:

1. $2x-3y=4$
$3x+2y=32.$

2. $5x-y=1$
$2x+3y=27.$

3. $x+2y=7$
$3x-2y=5.$

4. $7x+4y=2$
$5x-3y=19.$

5. $3x-2y=-2$
$x+y=6.$

6. $5x+7y=12$
$2x+21y=23.$

Solve the following problems:

7. The weight of water in pounds corresponding to the volume in cubic inches is given by the equation $w=\frac{62.4}{1728}v$. Represent the equation graphically. From the graph find the weights of 100 cubic inches, 200 cubic inches, 400 cubic inches of water. Find the volume of 10 pounds, 15 pounds, 25 pounds.

8. The rate at which sound travels at a given temperature is given by the formula $v=\frac{12648+13t}{12}$, where v is the number of feet a second and t is the number of degrees Fahrenheit. Make a graph of the equation and from it find the velocity at 32°; 72°.

9. The velocity of an object thrown downward is given by the formula $v=k+32t$ where k is the initial velocity. Make a graph of the equation when the initial velocity is 64 feet a second.

Historical note on systems of equations. The study of systems of equations (equations containing two or more unknowns) dates back to the time of Diophantus of Alexandria who lived near the middle of the third century after Christ. Practically nothing is known of his life. Among other works he wrote one on arithmetic which contains a considerable amount of algebra. The treatise is devoted to the solution of problems leading to equations of the first and second degree. Being several centuries ahead of his time in his knowledge of algebra, he stands out as one of the great mathematicians of Greek civilization, but his writings were not sufficiently known to influence the development of Greek and European mathematics. When his work finally became known in Europe in the 15th century, mathematics had passed beyond the stage to which it had been advanced by Diophantus.

René Descartes (1596–1650) had more influence on the development of mathematics during the 17th century than any other man of his time. He was a mathematical genius, a physicist, and philosopher. He is the inventor of the graphic treatment of equations by which algebra, arithmetic, and geometry were closely related to each other.

Algebraic Methods of Solving Equations with Two Unknowns

110. What is meant by elimination. In solving algebraically equations in two unknowns one may, by combining the equations, derive a single equation containing but one unknown. Thus, through the processes of combining the equations we *eliminate* (remove) one of the unknowns. The method to be used to bring about the elimination depends upon the form of the equations to be solved. Two methods will be explained in §§111 and 112.

111. Eliminating by substitution. As the name suggests, the method of eliminating one of the unknowns involves a substitution. For example, in the pair of equations $\begin{cases} 2x+3y=12 \\ y=2x-1 \end{cases}$

you may substitute $2x-1$ in place of y in the first equation and obtain $2x+3(2x-1)=12$, which contains only one unknown.

Although the method can be used for any pair of equations it is especially useful when one of the unknowns is easily expressed in terms of the other. The follcwing examples illustrate the process:

1. Solve by eliminating by substitution the system

$$8a-3b=30$$
$$b=a-5.$$

Solution: Substitute $a-5$ for b in the first equation.

$$\text{Then } 8a-3(a-5)=30.$$
$$\therefore 8a-3a+15=30.$$
$$5a=15.$$
$$a=3.$$
$$\therefore b=a-5=-2.$$
$$\therefore (a,\ b)=(3,\ -2).$$

Check the result by substituting 3 and -2 for a and b respectively in the original equations.

2. Solve the system of equations

$$3x-5y=1$$
$$2x=y+3.$$

Solution: Solving the second equation for x, you have

$$x=\frac{y+3}{2}.$$

Substitute $\frac{y+3}{2}$ in place of x in the first equation.

Then $3\left(\frac{y+3}{2}\right)-5y=1.$

Multiply both members by 2. This gives

$$3y+9-10y=2.$$
$$\therefore 7=7y.$$
$$\therefore y=1$$
$$\text{and } x=\frac{y+3}{2}=2.$$
$$\therefore (x,\ y)=(2,\ 1).$$

Check the result.

Exercises

Solve the following systems of equations:

1. $x+y=7$, $5x-2y=10.$
2. $8x-y=43$, $10x+3y=75.$
3. $10x-y=3$, $2x+2y=17.$
4. $x+4y=4$, $2x+2y=5.$
5. $2x+y=13$, $x+2y=14.$
6. $3a+b=11$, $5a-b=13.$
7. $4x=3y$, $2y=3x-1.$
8. $3a+2b=22$, $3a-b=25.$
9. $m-7n=0$, $3m-n=60.$
10. $a-3b=9$, $2a+7b=-6.$
11. $m-3n=0$, $4m+5n=38.$
12. $3x+5y=29$, $x+2y=11.$

In the following formulas eliminate the literal numbers as indicated:

13. $A=bh$, $h=4c$. Eliminate h.

14. $V=abc$, $4a=3h$. Eliminate a.

15. $s=\frac{1}{2}gt^2$, $v=gt$. Eliminate t.

16. $T=\pi r(h+r)$, $3r=2h$. Eliminate h.

17. $l=a+(n-1)d$, $s=\frac{n}{2}(2a+(n-1)d)$. Eliminate d.

18. $C=\frac{5}{9}(F-32)$, $F=3C$. Eliminate C.

19. $x^2+y^2=r^2$, $x=6y$. Eliminate y.

20. $A=\pi r^2$, $c=2\pi r$. Eliminate r.

21. $s=\frac{n}{2}(a+l)$, $l=a+(n-1)d$. Eliminate a.

22. $t=\pi\sqrt{\frac{l}{g}}$, $t=\frac{1}{2}$. Find l.

23. $m=\frac{r}{2}\sqrt{2+\sqrt{3}}$, $p=3r^2$. Eliminate r.

112. Eliminating by addition or subtraction. This method is used when the coefficients of the same unknown in both equations have the same numerical value, as in

$$3x+y=11$$
$$5x-y=13.$$

To eliminate y add the equations. This gives $8x=24$.

When the coefficients are not the same, multiply one equation by a number to make them the same.

Thus, in

$$5x+6y=-5$$
$$10x-9y=-6$$

multiply the first equation by 2.

This gives

$$10x+12y=-10$$
$$\underline{10x-\ \ 9y=-6\ }$$

By subtracting you have

$$21y=-4.$$

It may be necessary to multiply *both* equations to make the coefficients of one unknown the same.

Thus, in $\begin{aligned} 15x+\ 8y&=3 \\ 6x-12y&=5 \end{aligned}$ if you multiply the first equation by 2 and the second equation by 5 you have

$$\begin{aligned} 30x+16y&=6 \\ 30x-60y&=25. \end{aligned}$$

By subtraction you obtain $\quad 76y=-19.$

The following illustration shows the complete process:

Solve the system $\quad \begin{cases} 5x+9y=8 \\ 9x-6y=7. \end{cases}$

Solution: Examination of the coefficients shows that the elimination of y gives simpler resulting equations than the elimination of x.

Hence multiply the first equation by 2 and the second by 3.

This gives
$$\begin{aligned} 10x+18y&=16 \\ 27x-18y&=21. \end{aligned}$$

Adding you have $\quad 37x=37.$

$$\therefore x=1.$$

Substitute 1 for x in either of the given equations.

This gives
$$10+18y=16$$
$$\text{or } 18y=6.$$
$$\therefore y=\tfrac{1}{3}.$$
$\therefore (x,\ y)=(1,\ \tfrac{1}{3})$ is the required solution.

Check the solution by substituting in the original equations.

Exercises

Solve the following systems of equations:

1. $2a+3b=27$
$5a-2b=1.$

2. $2x+7y=52$
$3x-5y=16.$

3. $4a+7b=121$
$8a-3b=55.$

4. $2x+7y=-1$
$5x+8y=7.$

5. $5a+10b=14$
$2a+\ \ 5b=4.$

6. $2m-\ \ 6n=-9$
$3m-18n=10.$

7. $7a-\ \ 2b=3$
$6a-19b=-89.$

8. $7x-3y=2$
$3x+7y=5.$

9. $3m+4n=12$
$2m-5n=54.$

10. $2x-7y=-34$
$7x-2y=16.$

11. $3a+2b=7$
$2a+3b=8.$

12. $3a-4b=12$
$4a+3b=-6.$

13. $\frac{2}{x}+\frac{3}{y}=27$
$\frac{5}{x}-\frac{2}{y}=1.$

Solution: Multiply the first equation by 2 and the second by 3. This gives

$$\frac{4}{x}+\frac{6}{y}=54$$
$$\frac{15}{x}-\frac{6}{y}=3.$$

Add the two equations and show that

$$\frac{19}{x}=57$$
$$\text{or } 19=57x.$$
$$\therefore x=\tfrac{1}{3}.$$

Substitute $\frac{1}{3}$ for x in the first equation. Then

$$\frac{4}{\frac{1}{3}}+\frac{6}{y}=54.$$

$$12+\frac{6}{y}=54.$$

$$\frac{6}{y}=42.$$

$$6=42y.$$

$$y=\tfrac{1}{7}.$$

$$\therefore (x, y)=(\tfrac{1}{3}, \tfrac{1}{7}).$$

14. $\dfrac{1}{a}+\dfrac{1}{b}=25$
 $\dfrac{1}{a}-\dfrac{1}{b}=15.$

15. $\dfrac{6}{x}+\dfrac{12}{y}=-1$
 $\dfrac{8}{x}-\dfrac{9}{y}=7.$

16. $\dfrac{2}{a}+\dfrac{3}{b}=13$
 $\dfrac{5}{a}+\dfrac{1}{b}=13.$

17. $\dfrac{5}{x}+\dfrac{2}{y}=7$
 $\dfrac{3}{x}+\dfrac{2}{y}=5.$

If you are interested in another method of solving linear equations in two unknowns study §165.

Solving Linear Equations with Three Unknowns

113. How to solve equations in three unknowns. The algebraic methods for solving equations in two unknowns are also used to solve equations in three or more unknowns. The following examples show how to solve such systems:

1. Solve:
$$\begin{aligned} a+c&=-2\\ b+2c&=3\\ a+2b+3c&=4. \end{aligned}$$

Suggestion: The solution is simplified in this case by the fact that the first two equations contain only two unknowns. You may eliminate a from the first and third equations and obtain an equation in b and c, which together with the second equation will give the values of b and c. Or you may eliminate b from the second and third equations and use the resulting equation with the first.

Solution: Following the last suggestion write the third equation:

$$a+2b+3c=4.$$

Multiply the second equation by 2. Then

$$2b+4c=6.$$

Subtracting, you have the equation

$$a-c=-2.$$

The first equation is

$$a+c=-2.$$

$$\therefore 2a=-4.$$

$$a=-2$$

$$\text{and } c=0.$$

Substituting the values of a and c in the equation $a+2b+3c=4$ you find that

$$-2+2b=4.$$

$$\therefore 2b=6$$

$$\text{and } b=3.$$

$$\therefore (a, b, c)=(-2, 3, 0).$$

2. Solve the system:

$$x-y+z=2$$

$$x+y+z=4$$

$$2x+y+z=1.$$

Solution: Subtracting the second equation from the third you have

$$x = -3.$$

Subtracting the second equation from the first you have

$$-2y = -2.$$
$$\therefore y = 1.$$

Substituting the values of x and y in the third equation you have

$$-6 + 1 + z = 1.$$
$$\therefore z = 6.$$
$$\therefore (x, y, z) = (-3, 1, 6).$$

3. Solve the system:

$$\begin{aligned} 2x + 3y + 4z &= 16 \qquad (1) \\ 5x - 8y + 2z &= 1 \qquad (2) \\ 3x - \ \ y - 2z &= 5. \qquad (3) \end{aligned}$$

Solution: Add equations (2) and (3):

$$8x - 9y = 6. \qquad (4)$$

Write equation (1) and multiply equation (2) by 2:

$$\begin{array}{rrl} & 2x + 3y + 4z = 16 & (1) \\ & 10x - 16y + 4z = 2 & (5) \\ \hline \text{Subtract (5) from (1):} & -8x + 19y = 14 & (6) \\ \text{Write equation (4):} & 8x - 9y = 6 & (4) \\ \hline \text{Add:} & 10y = 20 & \\ & \therefore y = 2. & \end{array}$$

Substitute 2 for y in equation (4):

$$8x - 18 = 6$$
$$\therefore 8x = 24.$$
$$\therefore x = 3.$$

Substitute 3 for x and 2 for y in equation (1):

$$6+6+4z=16$$
$$4z=4$$
$$z=1$$
$$\therefore (x,\ y,\ z)=(3,\ 2,\ 1).$$

Check by substituting 3, 2, 1 in the original equations.

Exercises

Solve the following systems:

1. $a+b=4$
   $a+c=6$
   $b+c=8.$

2. $2x-y=11$
   $x-10z=-7$
   $x+y+z=-1.$

3. $2a+b=-1$
   $6a+4b+2c=1$
   $10a-15b+15c=57.$

4. $3a-b-c=1$
   $a+2b-2c=-3$
   $a-3b+c=2.$

5. $3a-3b+5c=12$
   $5a+2b-4c=-3$
   $4a+5b+2c=20.$

6. $2x-3y+3z=14$
   $2x+3y+z=16$
   $5x+4y-2z=15.$

7. $x-y+3z=0$
   $5x+2y+z=14$
   $2x+3y+4z=-14.$

8. $a-2b+c=1$
   $3a+b+c=14$
   $a+b+2c=15.$

Problems with Two or More Unknowns

114. When to use equations in several unknowns in solving problems. When a problem contains several unknowns which can be expressed easily in terms of one letter, an equation with *one* unknown should be used. For example, if you are to find the three angles of a triangle and if it is known that the first is twice as large as the second and the third $\frac{1}{3}$ as large as the second, you

can express the first and third angles easily in terms of the second angle. The equation is $2x+x+\frac{x}{3}=180$. Expressed in three unknowns the equations would be

$$x=2y$$
$$z=\tfrac{1}{3}y$$
$$x+y+z=180.$$

By processes of elimination you would have to derive a single equation in one unknown. Hence, it is simpler to use only one unknown.

However, when the relations between the unknowns are not simple it is advisable to use several unknowns and several equations.

Exercises

Solve the following problems:

1. A farmer wishes to get 10 gallons of 3.6% milk (milk yielding 3.6% butter fat) by mixing 3.2% milk and 4% milk. How much of each kind of milk shall he use?

2. How can I invest $3200 in two parts, one at 5% and one at 6%, yielding together a total income of $180 a year?

3. The sum of n terms of an arithmetical progression is given by the formula $s=\frac{n}{2}(a+l)$. The nth term l is given by the formula $l=a+(n-1)d$. Express s in terms of a and d.

4. Two partners in business have made a profit of $10,000 during a given year. They have an agreement that one of them is to receive $1000 more than two thirds of the amount received by the other. How much is each to receive?

5. Going downstream a steamer travels 60 miles in 3 hours. The return trip is made in 5 hours. Find the rate of the current and the rate at which the steamer can travel in still water.

6. Two men working together can build a fence in 2 days. After the first day one of the men quits work and the other finishes the work in 3 days. In how many days could each build the fence working alone?

7. The sum of three numbers is 2. The second is 6 more than the third and the sum of the first and three times the second is 4. Find the numbers.

8. A man has an opportunity to invest a sum of money in two different ways, one yielding 4% and the other 5%. What part of $10,000 should he invest at 4% and what part at 5% to obtain an annual income of $450?

9. A druggist wishes to obtain one quart of an 80% solution of ammonia by mixing a 90% solution with a 60% solution. How much must he take from each?

10. One side of a field is 10 rods long. It is to be divided so that the parts are in the ratio 2:3. How long is each part?

11. The perimeter of a rectangle is 283 feet and the adjacent sides are to each other as 3:8. How long is each side?

12. In an arithmetical progression the last term is 32 when 10 terms are taken, and 10 when 20 terms are taken. Find the first term and the common difference, using the formula $l = a + (n-1)d$.

13. In an arithmetical progression the sixteenth term is 40 and the sixth term is 10. Find the first term and the common difference.

Supplementary Topics

115. Fractional equations. In solving fractional equations it is usually best to remove the denominators

by multiplying every term of the equation by a number. Thus, in the equation $\frac{2x+6}{5}+\frac{y}{3}=8$, remove the denominators by first multiplying every term by 15.

This gives

$$\frac{\overset{3}{\cancel{15}}(2x+6)}{\cancel{5}}+\frac{\overset{5}{\cancel{15}}y}{\cancel{3}}=15\times 8, \text{ or } 6x+18+5y=120.$$

In some cases it is better to leave the fractions. For example, in the system of equations $\begin{cases}\frac{1}{x}+\frac{1}{y}=5\\ \frac{2}{x}+\frac{3}{y}=12\end{cases}$ x or y should be eliminated without first removing the denominators.

Exercises

Solve the following systems of equations:

1. $\frac{x+2}{y+2}=2$
   $\frac{x+7}{y+7}=\frac{3}{2}.$

2. $\frac{x+4}{4}-\frac{2y-4}{7}=1$
   $\frac{x-2}{3}+\frac{y-4}{5}=3.$

3. $\frac{3x-8}{9}=\frac{5y-3}{2}+31$
   $\frac{7x-1}{5}+\frac{3y+6}{10}+28=0.$

4. $\frac{3a-b}{2}+\frac{a+b}{3}=4$
   $\frac{3a+b}{11}+2=\frac{3a-b}{3}.$

5. $\frac{x}{4}-\frac{y+1}{3}=1$
   $\frac{x}{3}-\frac{3y+1}{4}=0.$

6. $\frac{1}{x-y}+\frac{1}{x+y}=15$
   $\frac{4}{x-y}-\frac{5}{x+y}=17.$

7. $\dfrac{3}{4x-y}-\dfrac{5}{2x-y}=2$

$\dfrac{4}{4x-y}+\dfrac{3}{2x-y}=-\frac{23}{5}.$

8. $\dfrac{8a-3b}{2}+6b+9=0$

$8a=3b+1.$

116. Equations with literal coefficients. The methods of §§111 and 112 are used in equations containing literal coefficients. The following two examples illustrate the process:

1. Solve the system of equations

$$\begin{aligned} x-y+1&=0\\ ax+by-c&=0.\end{aligned}$$

Solution: Solve the first equation for x. This gives

$$x=y-1.$$

Substitute $y-1$ for x in the second equation.

Then $a(y-1)+by-c=0$
or $ay-a+by-c=0.$

Collect the terms containing y:

$$(a+b)y=a+c$$

$$\therefore y=\frac{a+c}{a+b}$$

and $x=\dfrac{a+c}{a+b}-1=\dfrac{a+c-(a+b)}{a+b}=\dfrac{c-b}{a+b}.$

2. Solve the system of equations:

$$\begin{aligned} ax+by&=c\\ dx+ey&=f.\end{aligned}$$

Solution: Multiply the first equation by d and the second by a.

$$\begin{array}{rl} \text{Then} & adx+bdy=cd \\ \text{and} & \underline{adx+aey=af.} \\ \text{Subtract:} & bdy-aey=cd-af \\ \text{Collect terms:} & (bd-ae)y=cd-af \\ & y=\dfrac{cd-af}{bd-ae}. \end{array}$$

To find x, eliminate y, *i.e.*, multiply the first equation by e and the second by b.

$$\begin{array}{rl} \text{Then} & aex+bey=ce \\ & \underline{bdx+bey=bf.} \\ & \therefore (ae-bd)x=ce-bf \\ & \therefore x=\dfrac{ce-bf}{ae-bd}. \end{array}$$

Exercises

Solve the following systems of equations:

1. $ax+ay=b^2$
   $x-y=a.$

2. $ax+by=2ab$
   $y=x-a.$

3. $bx+ay=a+b$
   $abx-aby=a^2-b^2.$

4. $\dfrac{x-a}{y+b}=\dfrac{a}{b}$
   $\dfrac{x+a}{y-b}=\dfrac{a+1}{b}.$

5. $\dfrac{a}{x}+\dfrac{b}{y}=c$
   $\dfrac{b}{x}+\dfrac{a}{y}=d.$

6. $bx-ay=b^2$
   $(a-b)x+by=a^2.$

117. What every pupil should be able to do. The study of Chapter VIII should enable you to do the following:

1. To solve graphically two linear equations with two unknowns.
2. To solve algebraically two linear equations with two unknowns.
3. To solve algebraically three linear equations with three unknowns.
4. To solve problems leading to equations with several unknowns.

118. Typical problems and exercises. You should be able to solve the following problems and others of the same type:

1. Solve by graph:

$$5x+2y=34$$
$$7x-3y=7.$$

2. Solve algebraically:

$$3x+2y=23$$
$$2x+3y=27.$$

3. Solve:

$$2x+3y-4z=16$$
$$8x-3y-4z=28$$
$$4x-2y+3z=45.$$

4. Solve:

$$\frac{2}{x}+\frac{3}{y}=29$$
$$\frac{3}{x}+\frac{2}{y}=31.$$

5. A sum of $2800 has been invested in two parts, at 4.5% and 5% respectively, yielding $130.50 a year. Find each part.

CHAPTER IX

RADICALS

Fractional Exponents

119. What you have previously learned about exponents. In Chapter IV it was shown that exponents originally indicated products with several equal factors. Thus, a^3 means aaa and a^4 means $aaaa$. This gives meaning only to the case when exponents are whole numbers.

Later it was shown that expressions like x^0, x^{-4} in which the exponent is zero or a negative number have also definite interpretations. Thus, $x^0=1$ and $x^{-4}=\dfrac{1}{x^4}$. In this chapter the meaning of exponents will be extended to include not only positive and negative numbers, or zero, but also fractions.

120. The meaning of fractional exponents. To interpret the expression $a^{\frac{1}{2}}$ assume that the law $a^m \cdot a^n = a^{m+n}$ may be applied to $a^{\frac{1}{2}}$. Then $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a$.

This suggests that $a^{\frac{1}{2}}$ may be considered one of the two equal factors whose product is a, that is, that $a^{\frac{1}{2}}$ is but another symbol for expressing the number $\sqrt{a}$.

A root of a number indicated by a radical sign, as $\sqrt{a}$, is a *radical*.

If it is assumed that the law $a^m \cdot a^n = a^{m+n}$ may be applied to $a^{\frac{1}{3}}$, then $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$. This means that $a^{\frac{1}{3}}$ may be considered as one of the three equal factors whose product is a, or that $a^{\frac{1}{3}}$ and $\sqrt[3]{a}$ denote the same thing.

JOHN WALLIS

John Wallis (1616–1703) was professor of geometry at Oxford. He deserves credit for being among the first to extend the meaning of exponents so as to include negative and fractional cases. He wrote on many subjects relating to mechanics, physics, astronomy, and other fields in which mathematics is used. His main interest was in mathematics, especially arithmetic and algebra.

In general,

$$\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}.}$$

If you apply the multiplication law to $a^{\frac{2}{3}}$ and to $a^{\frac{3}{4}}$ you have

$$a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{6}{3}} = a^2$$

and

$$a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} = a^{\frac{12}{4}} = a^3.$$

Thus, $a^{\frac{2}{3}}$ may be defined as one of the three equal factors whose product is a^2, *i.e.*, $a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

Similarly, $a^{\frac{3}{4}}$ may be regarded as one of the four equal factors whose product is a^3, *i.e.*, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$.

From the preceding discussion it follows that the following meaning may be assigned to a fractional exponent:

In the power $a^{\frac{m}{n}}$ the denominator is the index of a root and the numerator is an exponent.

In symbols this may be expressed by the formula

$$\boldsymbol{a^{\frac{m}{n}} = \sqrt[n]{a^m}.}$$

EXERCISES

Express the following with radical signs:

1. $a^{\frac{1}{4}}$

2. $a^{\frac{1}{5}}$

3. $m^{\frac{3}{2}}$

4. $m^{\frac{3}{5}}$

5. $y^{\frac{5}{2}}$

6. $y^{\frac{5}{6}}$

7. $x^{\frac{3}{2}}y^{\frac{1}{4}}$

8. $a^{\frac{1}{2}}b^{\frac{1}{3}}$

9. $(3x)^{\frac{2}{3}}$

10. $(2x)^{\frac{3}{4}}$

Find the value of each of the expressions in Exercises 11 to 24:

11. $49^{\frac{1}{2}}$

Solution: $49^{\frac{1}{2}}=\sqrt{49}=7.$

12. $4^{\frac{3}{2}}$

Solution: $4^{\frac{3}{2}}=\sqrt{4^3}=8.$

13. $27^{\frac{1}{3}}$

14. $8^{\frac{2}{3}}$

15. $32^{\frac{3}{5}}$

16. $(.25)^{\frac{1}{2}}$

17. $(\frac{27}{8})^{\frac{1}{3}}$

18. $81^{\frac{3}{4}}$

19. $(-125)^{\frac{1}{3}}$

20. $(-64)^{\frac{1}{3}}$

21. $16^{\frac{3}{4}}$

22. $.008^{\frac{1}{3}}$

23. $(\frac{9}{16})^{\frac{3}{2}}$

24. $.125^{\frac{2}{3}}$

25. Show that $\sqrt[3]{x^2}=(\sqrt[3]{x})^2.$

Solution: $\sqrt[3]{x^2}=x^{\frac{2}{3}}=(x^{\frac{1}{3}})^2=(\sqrt[3]{x})^2.$

26. Show that $\sqrt[4]{x^3}=(\sqrt[4]{x})^3.$

Find the value of each of the following:

27. $(64)^{-\frac{1}{3}}$

Solution: $64^{-\frac{1}{3}}=\frac{1}{64^{\frac{1}{3}}}=\frac{1}{\sqrt[3]{64}}=\frac{1}{4}.$

28. $(27)^{-\frac{1}{3}}$

29. $(-32)^{-\frac{1}{5}}$

30. $81^{-\frac{1}{4}}$

31. $8^{-\frac{2}{3}}$

32. $27^{-\frac{2}{3}}$

33. $81^{-\frac{3}{4}}$

Changing a Radical to the Simplest Form

121. An important law for simplifying radicals. You have seen that $\sqrt{a}=a^{\frac{1}{2}}$, by §120,

and that $\sqrt{b}=b^{\frac{1}{2}}$.

$\therefore \sqrt{a}\ \sqrt{b}=a^{\frac{1}{2}}\cdot b^{\frac{1}{2}}=(ab)^{\frac{1}{2}}$, by §42,

$=\sqrt{ab}$, by §120.

Hence, $\boldsymbol{\sqrt{a}\ \sqrt{b}=\sqrt{ab}}$

or $\boldsymbol{\sqrt{ab}=\sqrt{a}\ \sqrt{b}}$.

Similarly, $\boldsymbol{\sqrt[n]{ab}=\sqrt[n]{a}\ \sqrt[n]{b}}$.

These laws may be used to change radicals to the simplest form. For example, $\sqrt{75}$ may be changed to: $\sqrt{25\times 3}=\sqrt{25}\ \sqrt{3}=5\sqrt{3}$.

To change $\sqrt[3]{81a^5b^6c^8}$ to the simplest form proceed as follows:

1. Factor the number under the radical sign finding all the factors that are cubes:

$$\sqrt[3]{81a^5b^6c^8}=\sqrt[3]{27\cdot 3a^3a^2b^6c^6c^2}.$$

2. Extract the cube roots of the factors that are cubes:

$$\sqrt[3]{27\cdot 3a^3a^2b^6c^6c^2}=3ab^2c^2\sqrt[3]{3a^2c^2}.$$

Exercises

Change each of the following to the simplest form:

1. $\sqrt{52}$	**6.** $\sqrt{96}$	**11.** $\sqrt{99a^2}$	**16.** $\sqrt[3]{54x^5y^3}$
2. $\sqrt{80}$	**7.** $\sqrt[4]{512}$	**12.** $\sqrt[3]{a^4b^3}$	**17.** $\sqrt[4]{100a^2x^8}$
3. $\sqrt[3]{24}$	**8.** $\sqrt{108}$	**13.** $\sqrt[3]{-27a^4b^3}$	**18.** $\sqrt[3]{64x^6y^5}$
4. $\sqrt{63}$	**9.** $\sqrt[3]{a^6}$	**14.** $\sqrt[4]{128a^6x^3}$	**19.** $\sqrt[3]{250a^2b^6}$
5. $\sqrt[3]{270}$	**10.** $\sqrt{3r^2}$	**15.** $\sqrt{8a^2x^3y^5}$	**20.** $\sqrt[6]{64x^{10}y^{18}}$

21. $\sqrt{a^2b^2+a^3b^4}$

Solution: 1. Factor the number under the radical sign.

Thus, $\sqrt{a^2b^2+a^3b^4}=\sqrt{a^2b^2(1+ab^2)}$.

2. Take the square root of the factors. This gives

$$ab\sqrt{1+ab^2}.$$

$$\therefore \sqrt{a^2b^2+a^3b^4}=ab\sqrt{1+ab^2}.$$

22. $\sqrt{9ax^2-9x^3}$

23. $\sqrt[3]{8(a+b)^2x}$

24. $\sqrt{(x+y)\,(x^2-y^2)}$

25. $\sqrt{(a^2-b^2)\,(a-b)}$

122. How to change a radical to one of lower order. When the index of the root and the exponent of a factor under the radical sign have a common factor, as $\sqrt[6]{a^4}$, it is possible to change the radical to a lower order. Show that $\sqrt[6]{a^4}=(a^4)^{\frac{1}{6}}=a^{\frac{4}{6}}=a^{\frac{2}{3}}=\sqrt[3]{a^2}$.

Similarly show that

$$\sqrt[4]{9a^8b^2}=(3^2a^8b^2)^{\frac{1}{4}}=3^{\frac{2}{4}}a^{\frac{8}{4}}b^{\frac{2}{4}}=3^{\frac{1}{2}}a^2b^{\frac{1}{2}}=a^2\sqrt{3b}.$$

Exercises

Change the following to radicals of lower order:

1. $\sqrt[4]{25}$ **5.** $\sqrt[8]{16}$ **9.** $\sqrt[10]{64}$ **13.** $\sqrt[6]{729a^9}$

2. $\sqrt[6]{216}$ **6.** $\sqrt[6]{y^9}$ **10.** $\sqrt[6]{49}$ **14.** $\sqrt[4]{16a^4x^2y^2}$

3. $\sqrt[6]{8}$ **7.** $\sqrt[6]{192}$ **11.** $\sqrt[6]{72}$ **15.** $\sqrt[4]{80x^6y^{12}}$

4. $\sqrt[8]{625}$ **8.** $\sqrt[8]{243}$ **12.** $\sqrt[6]{27y^3}$ **16.** $\sqrt[10]{32a^5b^{10}c^{15}}$

123. How to simplify a root of a fraction. The examples below show how a root of a fraction may be

changed to the simplest form. The following law is used in the process:

$$\sqrt{\frac{a}{b}}=\left(\frac{a}{b}\right)^{\frac{1}{2}}=\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}=\frac{\sqrt{a}}{\sqrt{b}}.$$

This may be written briefly:

$$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}.$$

1. Simplify $\sqrt{\frac{9c^4h}{16}}$.

Solution: $\sqrt{\frac{9c^4h}{16}}=\frac{\sqrt{9c^4h}}{\sqrt{16}}=\frac{3c^2\sqrt{h}}{4}.$

2. Simplify $\sqrt{\frac{3}{5}}$.

Solution: Multiply numerator and denominator under the radical sign by 5 to make the denominator a square.

Then $\sqrt{\frac{3}{5}}=\sqrt{\frac{3\cdot5}{5\cdot5}}=\frac{1}{5}\sqrt{15}.$

3. Simplify $\sqrt{\frac{8a^2}{27b}}$.

Solution: $\sqrt{\frac{8a^2}{27b}}=\sqrt{\frac{4\cdot2a^2}{9\cdot3b}}=\frac{2a}{3}\sqrt{\frac{2}{3b}}.$

To make the denominator a perfect square multiply numerator and denominator under the radical sign by $3b$. Then extract the root. This gives

$$\frac{2a}{3}\sqrt{\frac{2\cdot3b}{3b\cdot3b}}=\frac{2a}{3\cdot3b}\sqrt{6b}=\frac{2a}{9b}\sqrt{6b}.$$

Exercises

Simplify the following:

1. $10\sqrt{\frac{3}{5}}$
2. $\sqrt[3]{\frac{1}{4}}$
3. $\sqrt{\frac{4a}{5b}}$
4. $\sqrt[3]{\frac{3}{4}}$
5. $\sqrt[3]{\frac{8}{9}}$
6. $\sqrt[4]{\frac{8}{27}}$
7. $\sqrt[4]{\frac{3}{8}}$
8. $\sqrt[3]{\frac{27}{5}}$
9. $\sqrt[3]{\frac{9}{a^2}}$
10. $\sqrt[3]{\frac{1}{3m^2}}$
11. $\sqrt{\frac{x}{y^2z}}$
12. $\sqrt{\frac{5x^3}{3a^2y}}$
13. $\sqrt[3]{\frac{81}{4x^4}}$
14. $\sqrt[3]{\frac{4x^2}{25y^5}}$
15. $\sqrt[3]{\frac{5ax^2}{8bc^3}}$
16. $\sqrt[3]{\frac{8a^3c^4}{25b^2}}$
17. $\sqrt{\frac{1+x}{1-x}}$
18. $\sqrt{\frac{a+b}{a-b}}$

124. Changing a fraction having a radical in the denominator into one whose denominator contains no radical. Two cases will be considered here:

1. Fractions with denominators that are single radicals, as $\frac{2}{\sqrt{3}}$.

2. Fractions whose denominators are binomials containing a radical as one term, as $\frac{3}{2-\sqrt{5}}$.

The process of obtaining denominators without radicals is called *rationalizing* the denominator. The following examples explain the method:

(1) Rationalize the denominator of $\frac{\sqrt{2}}{\sqrt{5}}$.

Solution: Multiply numerator and denominator by $\sqrt{5}$, using §121.

Then $\dfrac{\sqrt{2}}{\sqrt{5}}=\dfrac{\sqrt{2}\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}=\dfrac{\sqrt{10}}{5}=\dfrac{1}{5}\sqrt{10}.$

(2) Rationalize the denominator of $\dfrac{1}{3+\sqrt{2}}\cdot$

Solution: Multiply numerator and denominator by $3-\sqrt{2}$.

Then $\dfrac{1}{3+\sqrt{2}}=\dfrac{3-\sqrt{2}}{(3+\sqrt{2})(3-\sqrt{2})}=\dfrac{3-\sqrt{2}}{3^2-(\sqrt{2})^2}=\dfrac{3-\sqrt{2}}{7}.$

(3) Rationalize the denominator of $\dfrac{2\sqrt{2}}{\sqrt{3}-\sqrt{2}}\cdot$

Solution: Multiply numerator and denominator by $\sqrt{3}+\sqrt{2}$.

Then $\dfrac{2\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\dfrac{2\sqrt{2}(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}=\dfrac{2\sqrt{6}+4}{3-2}=2\sqrt{6}+4.$

Exercises

Rationalize the denominators of the following fractions:

1. $\dfrac{\sqrt{3}}{\sqrt{5}}$
2. $\dfrac{\sqrt{a}}{\sqrt{b}}$
3. $\dfrac{3}{\sqrt{2}}$
4. $\dfrac{3\sqrt{3}}{5\sqrt{5}}$
5. $\dfrac{1}{\sqrt{5x}}$
6. $\dfrac{3}{5\sqrt{2x}}$
7. $\dfrac{\sqrt{5}}{\sqrt{3x}}$
8. $\dfrac{1}{\sqrt[3]{2}}$
9. $\dfrac{10}{\sqrt[3]{9}}$
10. $\dfrac{5}{3\sqrt{2x}}$
11. $\dfrac{8}{3\sqrt[3]{2y}}$
12. $\dfrac{5-\sqrt{2}}{4\sqrt{3}}$

13. $\dfrac{3+\sqrt{2}}{4-\sqrt{2}}$. **15.** $\dfrac{3-\sqrt{5}}{2+\sqrt{5}}$. **17.** $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$.

14. $\dfrac{1+\sqrt{3}}{3+\sqrt{3}}$. **16.** $\dfrac{2+3\sqrt{2}}{3-5\sqrt{2}}$. **18.** $\dfrac{5\sqrt{2}+4}{3\sqrt{2}-4}$.

19. $\tan 30° = \dfrac{1}{\sqrt{3}}$ (Fig. 39). Rationalize the denominator.

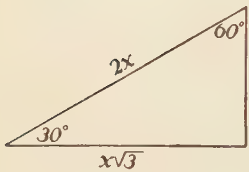


Fig. 39

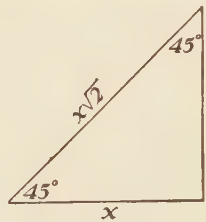


Fig. 40

20. $\sin 45° = \dfrac{1}{\sqrt{2}}$ (Fig. 40). Rationalize the denominator.

21. The radius of the circle circumscribed about an equilateral triangle (Fig. 41) is found as follows:

$$r^2 = \frac{r^2}{4} + \frac{a^2}{4}$$
$$4r^2 = r^2 + a^2$$
$$3r^2 = a^2$$
$$r^2 = \frac{a^2}{3}$$
$$r = \frac{a}{\sqrt{3}}.$$

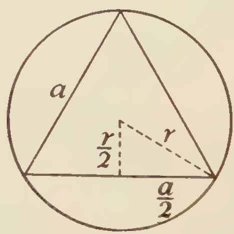


Fig. 41

Rationalize the denominator.

The Operations with Radicals

125. Adding and subtracting radicals. Two radicals, as $2\sqrt{5}$ and $3\sqrt{5}$, can be added in the same way as the similar terms $2x$ and $3x$ are added, *i.e.*, the coeffi-

cients of the common factor are added, and the sum is then multiplied by the common factor. It is necessary to change all radicals to the simplest form before combining them. The following examples illustrate the method:

1. Add and subtract as indicated:

$$3\sqrt{98}-7\sqrt{72}-3\sqrt{18}.$$

Solution:

$$\begin{aligned} 3\sqrt{98}-7\sqrt{72}-3\sqrt{18} &= 3\sqrt{2\cdot 49}-7\sqrt{2\cdot 36}-3\sqrt{2\cdot 9} \\ &= 21\sqrt{2}-42\sqrt{2}-9\sqrt{2} \\ &= -30\sqrt{2}. \end{aligned}$$

2. Add and subtract as indicated:

$$3\sqrt{5}+2\sqrt{\tfrac{4}{5}}-5\sqrt{\tfrac{1}{5}}.$$

Solution: As in example 1, first simplify each radical and then collect the similar terms:

$$\begin{aligned} 3\sqrt{5}+2\sqrt{\tfrac{4}{5}}-5\sqrt{\tfrac{1}{5}} &= 3\sqrt{5}+2\sqrt{\frac{4\cdot 5}{25}}-5\sqrt{\frac{5}{25}} \\ &= 3\sqrt{5}+\tfrac{4}{5}\sqrt{5}-\tfrac{5}{5}\sqrt{5} \\ &= 2\tfrac{4}{5}\sqrt{5}. \end{aligned}$$

EXERCISES

Add and subtract as indicated:

1. $\sqrt{50}+\sqrt{98}-\sqrt{32}$
2. $8\sqrt{2}+6\sqrt{8}-5\sqrt{32}$
3. $5\sqrt{27}-2\sqrt{192}+3\sqrt{300}$
4. $\sqrt[3]{16}-3\sqrt[3]{2}+\sqrt[3]{54}$
5. $8\sqrt{45}-\frac{3}{4}\sqrt{80}+3\sqrt{63}$
6. $2\sqrt[3]{16x^3}-7\sqrt[3]{54x^3}+\sqrt[3]{250x^3}$
7. $5\sqrt{\frac{6}{5}}+2\sqrt{\frac{15}{2}}-3\sqrt{\frac{3}{10}}$
8. $\sqrt{\frac{4}{6}}-6\sqrt{\frac{4}{3}}+3\sqrt{\frac{1}{3}}$
9. $6\sqrt{\frac{2}{7}}-4\sqrt{\frac{1}{14}}+3\sqrt{\frac{7}{2}}$
10. $2\sqrt{63}+\frac{3}{5}\sqrt{45}-3\sqrt{\frac{1}{5}}$
11. $\sqrt[3]{128}-3\sqrt[3]{81}+2\sqrt[3]{\frac{1}{4}}$
12. $3\sqrt{4\frac{1}{6}}-8\sqrt{\frac{4}{5}}+2\sqrt{3\frac{1}{5}}$

13. $\sqrt{108}-\sqrt{54}+\sqrt{300}-\sqrt{162}$

14. $2\sqrt[3]{\frac{1}{36}}-\frac{5}{2}\sqrt[3]{\frac{2}{9}}+\sqrt[3]{48}$

15. $\sqrt{32x^2}-\sqrt{8x^2}+\sqrt{18x^2}$

16. $\sqrt{\frac{m}{a^2}}+\sqrt{\frac{m}{b^2}}+\sqrt{\frac{m}{c^2}}$

17. $\sqrt[3]{2a^3x}+a\sqrt[3]{128x}-3a\sqrt[3]{250x}$

18. $\sqrt{\frac{1}{2}}+\sqrt{\frac{25}{2}}+\sqrt{\frac{1}{8}}+\sqrt{\frac{9}{8}}$

19. $6\sqrt{45}-3\sqrt{28}-\frac{3}{4}\sqrt{80}$

20. $2\sqrt{6}+5\sqrt{24}-\frac{3}{2}\sqrt{\frac{8}{3}}+2\sqrt{\frac{2}{3}}$

126. Multiplying radicals. The method of multiplying two radicals of the same order may be seen from the law $\sqrt{a}\,\sqrt{b}=\sqrt{ab}$ (§121).

Before multiplying change the radicals to the simplest form.

Exercises

Multiply as indicated:

1. $\sqrt{3}\,\sqrt{27}$

Solution: Simplifying the second radical you have

$$\sqrt{3}\times 3\sqrt{3}.$$

$$\therefore \sqrt{3}\,\sqrt{27}=\sqrt{3}\times 3\sqrt{3}=3\times 3=9.$$

2. $\sqrt{12}\,\sqrt{18}$

3. $2\sqrt{5}\,\sqrt{15}$

4. $4\sqrt{5}\,\sqrt{5}$

5. $\sqrt{a}\,\sqrt{ab^2}$

6. $\sqrt[3]{20}\,\sqrt[3]{12}$

7. $\sqrt[3]{9}\,\sqrt[3]{9}$

8. $\sqrt{5}\,\sqrt{35}$

9. $2\sqrt[4]{4}\cdot 6\sqrt[4]{8}$

10. $\sqrt{\frac{15}{16}}\cdot\sqrt{\frac{5}{8}}$

11. $2x\sqrt{x^3y}\cdot x^3\sqrt{xy^3}$

12. $4\sqrt[5]{a^6y^4}\cdot\sqrt[5]{ay^5}$

13. $3\sqrt{3a^2}\cdot 2\sqrt{15a}$

14. $(2\sqrt{2}-\sqrt{5}+3\sqrt{3})\sqrt{3}$

Solution: Multiply each term by $\sqrt{3}$. This gives

$$2\sqrt{6}-\sqrt{15}+9.$$

15. $(\sqrt{2}-\sqrt{5})(3\sqrt{5}+\sqrt{3})$

Solution: Multiply each term of the first binomial by each term of the second:

$$\begin{aligned}(\sqrt{2}-\sqrt{5})(3\sqrt{5}+\sqrt{3}) &= 3\sqrt{10}-3\times 5+\sqrt{6}-\sqrt{15}\\ &= 3\sqrt{10}-15+\sqrt{6}-\sqrt{15}.\end{aligned}$$

16. $(\sqrt{3}+6)(\sqrt{3}-4)$

17. $(\sqrt{2}+7)(\sqrt{2}-7)$

18. $(2\sqrt{3}+8)(2\sqrt{3}-8)$

19. $(\sqrt{2}-5)(3\sqrt{2}-4)$

20. $(2+\sqrt{5})^2$

21. $(2-5\sqrt{7})(2+5\sqrt{7})$

22. $(3\sqrt{2}-\sqrt{5})(4\sqrt{3}+\sqrt{7})$

23. $\dfrac{3+\sqrt{5}}{2}\cdot\dfrac{3-\sqrt{5}}{2}$

24. $(2\sqrt{a}+4\sqrt{c})(\sqrt{a}+3\sqrt{c})$

25. $(3\sqrt{30}+2\sqrt{5})(2\sqrt{5}+3\sqrt{6})$

26. $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$

27. $(\sqrt{x-3}+2)^2$

28. $\sqrt{3-\sqrt{5}}\ \sqrt{3+\sqrt{5}}$

29. $(\sqrt{x+2}-1)(\sqrt{x+2}+1)$

30. $(\sqrt{5}-3\sqrt{2})(3\sqrt{2}+\sqrt{5})(4\sqrt{3}-\sqrt{2})$

31. $(3\sqrt{2a}+5\sqrt{a-1})(3\sqrt{2a}-5\sqrt{a-1})$

Equations Involving Radicals

127. What is meant by an irrational equation. If an equation is written in its simplest form, and if the unknown appears under the radical sign, it is called an *irrational* equation. This includes the case where the unknown number has a fractional exponent. The equations $x-x^{\frac{1}{2}}+12=0$ and $\sqrt{3x-5}-4=0$ are illustrations of irrational equations.

128. How to solve irrational equations. The following examples explain a method of solving irrational equations:

1. Solve the equation $\sqrt{x+2}-4=0$.

Solution: Add 4 to each side of the equation. This leaves the radical alone on one side:

$$\sqrt{x+2}=4.$$

Square both sides of the equation. This gives

$$x+2=16$$
$$\therefore x=14.$$

Check: Left member	Right member
$\sqrt{14+2}$	4
$\sqrt{16}$	4
4	4

2. Solve the equation $x-1-\sqrt{x^2-5}=0$.

Solution: Add $\sqrt{x^2-5}$ to both members. This leaves the radical alone on one side of the equation:

$$x-1=\sqrt{x^2-5}.$$

Squaring both members you have

$$x^2-2x+1=x^2-5$$
$$\therefore -2x+1=-5$$
$$2x=6$$
$$x=3.$$

Check by substituting 3 for x in the original equation.

Exercises

Solve the following equations:

1. $\sqrt{x^2+7}=x+1.$
2. $\sqrt{2x+10}=\sqrt{3x+7}.$
3. $\sqrt{x-5}=5-\sqrt{x}.$
4. $\sqrt{9x^2-5}-3x-1=0.$
5. $\sqrt{x}-4+\sqrt{x+8}=0.$
6. $\sqrt{4x+3}=2\sqrt{x-1}+1.$
7. $\sqrt{x+16}+\sqrt{x}=8.$
8. $\sqrt{x-1}-\sqrt{x-4}-1=0.$
9. $2\sqrt{3a-5}=3\sqrt{a+1}.$
10. $\sqrt{y}-2=\sqrt{y-12}.$
11. $\sqrt{x+4}=3-\sqrt{x}.$
12. $\sqrt{a+6}-5=\sqrt{a+11}.$

13. The first term of a geometric progression of 7 terms is -2, and the last term is -128. Find the common ratio.

14. The first term of a geometric progression is 2, the common ratio is 2, and the sum of the terms is 6. Find the number of terms.

15. Between 8 and $\frac{1}{2}$ insert three numbers forming a geometric progression with 8 and $\frac{1}{2}$.

Suggestion: Let $a=8$, $l=\frac{1}{2}$, and $n=5$.

Find r and the required numbers. The required numbers are the *geometric means.*

16. Insert 5 geometric means between 3 and $\frac{1}{243}$.

17. The perpendicular drawn to the hypotenuse of a right triangle from the vertex of the right angle is the mean proportional between the segments of the hypotenuse. Find the perpendicular when the segments of the hypotenuse are 8 inches and 32 inches.

18. Find the side of an equilateral triangle whose area is 62.35 square yards.

19. Insert 4 geometric means between 3 and 96.

Supplementary Exercises and Topics

129. Exercises in radicals and fractional exponents. The following exercises give practice in the use of the laws of exponents and radicals. In most cases several laws apply, and you must select the one which leads to the result in the simplest way.

Exercises

Simplify the following:

1. $\dfrac{a^0x^{-3}y^{\frac{1}{2}}}{a^2x^{-2}y^{-\frac{3}{2}}}$

Solution: Divide numerator and denominator by $x^{-3}y^{-\frac{3}{2}}$.

Then $\dfrac{a^0x^{-3}y^{\frac{1}{2}}}{a^2x^{-2}y^{-\frac{3}{2}}}=\dfrac{y^{\frac{1}{2}+\frac{3}{2}}}{a^2x^{-2+3}}=\dfrac{y^{\frac{4}{2}}}{a^2x}=\dfrac{y^2}{a^2x}$.

2. $\dfrac{2^{\frac{1}{3}}x^{\frac{2}{3}}y^{\frac{1}{3}}}{2^{\frac{1}{5}}x^{-\frac{1}{5}}y^{\frac{2}{5}}}$

3. $\dfrac{81^{\frac{1}{3}}x^{\frac{2}{3}}y^{\frac{1}{3}}}{9^{\frac{1}{6}}x^{-\frac{1}{3}}y^{\frac{1}{6}}}$

4. $\dfrac{x^{-2}y}{x^{\frac{3}{2}}y^{-\frac{3}{2}}}$

5. $\dfrac{ab^{\frac{1}{2}}c^{-\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{3}{2}}c^{\frac{2}{3}}}$

6. Simplify $\sqrt{4x^{-4}y^{\frac{1}{2}}z^{-\frac{3}{2}}}$.

Solution: Change the radical sign into a fractional exponent and by means of the law $a^{-n}=\dfrac{1}{a^n}$, change all negative exponents into positive exponents:

$$\text{Then } \sqrt{4x^{-4}y^{\frac{1}{2}}z^{-\frac{3}{2}}}=4^{\frac{1}{2}}\left(\frac{1}{x^4}\right)^{\frac{1}{2}}(y^{\frac{1}{2}})^{\frac{1}{2}}\left(\frac{1}{z^{\frac{3}{2}}}\right)^{\frac{1}{2}}$$

$$=\sqrt{4}\cdot\frac{1}{x^2}\cdot y^{\frac{1}{4}}\cdot\frac{1}{z^{\frac{3}{4}}}$$

$$=\frac{2y^{\frac{1}{4}}}{x^2z^{\frac{3}{4}}}.$$

7. $\sqrt{\frac{1}{9}a^{-1}b^{-\frac{1}{2}}}$

8. $(ayx^{-1})^{\frac{1}{2}}\,(bxy^{-2})^{\frac{1}{3}}$

9. $\left(\sqrt[4]{16a^4b^8}\right)^{-2}$

10. $\left((27a^3)^{-\frac{1}{3}}\cdot\dfrac{1}{5a^{-4}}\right)^{-2}$

11. $\left(\dfrac{a^{\frac{2}{3}}\cdot 4b^{\frac{1}{8}}}{2^{\frac{2}{3}}b^{-\frac{9}{8}}}\right)^{\frac{3}{2}}$

12. $2^{\frac{1}{3}}\sqrt[6]{\frac{1}{3}}\cdot 3^{\frac{1}{6}}$

13. $\sqrt[7]{x^2y^{12}}\left(\dfrac{1}{xy}\right)^{\frac{1}{7}}\left(\dfrac{y^2}{x^3}\right)^{-\frac{2}{7}}$

14. $\dfrac{\sqrt[5]{3}}{3^{-\frac{4}{5}}}-\sqrt{3\cdot 27^{-1}}$

15. Expand $(\sqrt{x}+\sqrt{y})^4$.

Suggestion: Change $\sqrt{x}+\sqrt{y}$ to $x^{\frac{1}{2}}+y^{\frac{1}{2}}$ and use the binomial theorem.

16. Expand $(a^{-3}+b^{-3})^4$.

17. Expand $\left(\dfrac{\sqrt{x}}{y}+\dfrac{\sqrt{y}}{x}\right)^3$.

130. How to extract the square root of a polynomial. The process of extracting the square root of a polynomial is the same as that of extracting the square root of an arithmetical number. The following examples illustrate the method:

1. Extract the square root of x^2+6x+9.

Solution: (1) To find the first term of the root extract the square root of the first term of x^2+6x+9. This gives x.

$$
\begin{array}{r|l}
 & \quad x\ +3 \\
 & \overline{\sqrt{x^2+6x+9}} \\
 & \quad x^2 \\
\hline
2x+3 & \qquad 6x+9 \\
 & \qquad \underline{6x+9}
\end{array}
$$

(2) Write x over the radical sign, square it, and subtract the result from x^2+6x+9. This gives $6x+9$.

(3) To find the second term of the root, double the first term, divide this into $6x$, the first term of the remainder. This gives 3. Write the 3 to the right of the first term of the root over the radical sign.

(4) Write "twice the first term," when found in step 3, to the left of the remainder $6x+9$. Then write the second term of the root to the right of it, which gives $2x+3$. Multiply $2x+3$ by the second term of the root and write the product under the remainder $6x+9$. Subtract. If the remainder is zero the exact root has been found.

2. Extract the square root of

$$10x^2+1+12x^3+9x^4+4x.$$

Solution: (1) Arrange the terms of the polynomial according to descending powers of x, as

$$9x^4+12x^3+10x^2+4x+1.$$

$$\begin{array}{ll}
 & \quad 3x^2+2x \\
 & \sqrt{9x^4+12x^3+10x^2+4x+1} \\
 & \quad 9x^4 \\
\hline
6x^2+2x & \qquad \left| \begin{array}{l} 12x^3+10x^2+4x+1 \\ 12x^3+\ \ 4x^2 \\ \hline \end{array} \right. \\
6x^2+4x+1 & \qquad\qquad \left| \begin{array}{l} 6x^2+4x+1 \\ 6x^2+4x+1 \\ \hline \end{array} \right.
\end{array}$$

(2) Take the square root of $9x^4$, write it above the radical sign, square it, and subtract the result from the polynomial.

(3) Place twice $3x^2$ to the left of the remainder, divide it into $12x^3$, and write the result over the radical sign and to the right of $6x^2$.

Multiply $6x^2+2x$ by $2x$ and subtract the product from the remainder. This gives $6x^2+4x+1$.

(4) Double the $3x^2+2x$, place it to the left of the last remainder. Divide the first term into the first term of the remainder, and place the result over the radical sign and to the right of $6x^2+4x$. Multiply $6x^2+4x+1$ by 1 and subtract.

The preceding solution may be arranged as follows:

$$
\begin{array}{llll}
 & & & 3x^2+\ 2x \\
 & & & \overline{\sqrt{9x^4+12x^3+10x^2+4x+1}} \\
\sqrt{9x^4}=3x^2 & (3x^2)^2= & & 9x^4 \\
\text{Subtract: } 2(3x^2) & =6x^2 & & \overline{12x^3+10x^2+4x+1} \\
 & \dfrac{12x^3}{6x^2} & =2x & \\
 & (6x^2+2x)2x= & & 12x^3+\ 4x^2 \\
 & 2(3x^2+2x) & =6x^2+4x & \overline{\qquad\quad 6x^2+4x+1} \\
 & \dfrac{6x^2}{6x^2} & =1 & \\
 & (6x^2+4x+1)1= & & \qquad\quad 6x^2+4x+1 \\
\end{array}
$$

Exercises

Extract the square root of each of the following polynomials:

1. $x^4-4x^3+6x^2-4x+1$.
2. $30x+19x^2+x^4+6x^3+25$.
3. $4x^3+x^4-2x^2+9-12x$.
4. $16x^6+10x+1-8x^3+25x^2-40x^4$.
5. $16x^6+25x^4-20x^3-24x^5+10x^2-4x+1$.
6. $8a^3-4a-16a^4+16a^6+4a^2+1$.

7. $x^6-10x^4-4x^3+25x^2+20x+4.$

8. $\frac{9}{4}x^4-2x^3+\frac{67}{9}x^2-\frac{28}{9}x+\frac{49}{9}.$

9. $\frac{a^4}{4}-\frac{1}{3}a^3-\frac{41}{36}a^2+\frac{5}{6}a+\frac{25}{16}.$

10. $9x^6+12x^5-26x^4-14x^3+29x^2-10x+1.$

131. What every pupil should be able to do. After studying Chapter IX, you should be able to do the following:

1. To apply correctly, in problems involving radicals and exponents, the laws below:

$$a^{\frac{1}{n}}=\sqrt[n]{a}$$
$$a^{\frac{m}{n}}=\sqrt[n]{a^m}=\left(\sqrt[n]{a}\right)^m$$
$$\sqrt{a}\sqrt{b}=\sqrt{ab}$$
$$\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}.$$

2. To change radicals to the simplest form.
3. To add, subtract, multiply, and divide radicals.
4. To solve certain simple irrational equations.

132. Typical problems and exercises. The exercises below are typical of the work of Chapter IX. You should be able to work them and others similar to them.

1. Change $x^{\frac{3}{2}}y^{\frac{1}{5}}$ to the radical form.

2. Find the value of: $(-32)^{\frac{1}{5}}$; $8^{-\frac{2}{3}}$.

3. Change to the simplest form: $\sqrt[3]{64x^6y^5}$; $\sqrt{9ax^2+9x^3}$.

4. Rationalize: $\dfrac{3}{\sqrt{2}}$; $\dfrac{1+\sqrt{3}}{2-\sqrt{3}}$.

5. Add and subtract as indicated: $2\sqrt{63}+\frac{3}{5}\sqrt{45}-3\sqrt{\frac{1}{5}}$.

6. Multiply: $(3\sqrt{2}-\sqrt{5})(\sqrt{3}+2\sqrt{7})$.

7. Solve for x: $\sqrt{x+8}+\sqrt{x}=4$, and check the result.

CHAPTER X

A STUDY OF QUADRATIC EQUATIONS

A Review of Methods of Solving Quadratic Equations

133. A very simple quadratic equation. In this chapter you will first review briefly what you have learned about the quadratic equation in previous courses. The study will then be extended by developing a new method of solving quadratic equations and by discovering some important properties of the roots.

The simplest quadratic equation is of the form $ax^2=b$. Show that the equations $x^2=25$ and $2x^2-30=0$ are of that form. For each state the value of a and b.

To solve the equation $2x^2-30=0$, add 30 to both sides, divide both sides by 2, and extract the square root of each member of the resulting equation. Thus, you will have the following solution:

$$2x^2-30=0.$$
$$2x^2=30.$$
$$x^2=15.$$
$$x=\pm\sqrt{15}.$$

Note the positive and negative square root of 15. Check the results by substituting $+\sqrt{15}$ and $-\sqrt{15}$ in the original equation.

Exercises

Solve the following equations and check each:

1. $x^2-36=0$.
2. $2x^2-98=0$.
3. $6a^2-216=0$.
4. $5m^2-125=0$.
5. $3x^2=25$.

Suggestion: After finding the value of x, rationalize the denominator.

6. $2x^2-81=0$.
7. $5x^2-36=0$.
8. $4a^2-242=0$.
9. $6x^2-200=0$.

Solve the following equations as indicated:

10. $s=\frac{1}{2}gt^2$ for t.
11. $E=\frac{1}{2}mv^2$ for v.
12. $F=\dfrac{mM}{d^2}$ for d.
13. $f=\dfrac{mv^2}{R}$ for v.

134. Solving quadratic equations by factoring. When all the terms of a quadratic equation have been brought to the same side, and when the similar terms have been combined, they may form two or three terms. Thus, we have equations of the form $ax^2=b$, or of the form $ax^2+bx=0$, or of the form $ax^2+bx+c=0$. In all cases the factoring method (§74) is worth trying because it is easily carried out. The following examples show how it is used for each of the three forms:

1. Solve the equation $3x^2=75$.

Solution: Subtract 75 from both members. This gives $$3x^2-75=0.$$
Factor the left member: $3(x-5)(x+5)=0$.

The last equation is satisfied if $x-5=0$

or if $x+5=0$.

$$\therefore x_1=5$$
and $x_2=-5$.

2. Solve the equation $5x^2+15x=0$.

Solution: Factor the left side: $5x(x+3)=0$.

This equation is satisfied if $x=0$

or if $x+3=0$.

$\therefore x_1=0$

and $x_2=-3$.

3. Solve the equation $2x^2-5x-12=0$.

Solution: Factor the left side: $(2x+3)(x-4)=0$.

Put each factor equal to zero, *i.e.*, let $2x+3=0$

and $x-4=0$.

$\therefore x_1=-\frac{3}{2}$

and $x_2=4$.

Exercises

Solve the following equations and check the results:

1. $2x^2+5x=0$.
2. $3x^2+3=0$.
3. $6x^2-8x=0$.
4. $x^2-3x+2=0$.
5. $x^2-26x+25=0$.
6. $x^2-5x+6=0$.
7. $2x^2-3x-5=0$.
8. $2a^2-3a=35$.
9. $6m^2+7m+2=0$.
10. $x^2+8x=0$.

135. Solving quadratic equations by completing the square. Not every quadratic equation factors easily. The method by completing the square, however, may be used to solve any quadratic equation. It is reviewed in the following example:

Solve: $2x^2-3x-35=0$.

Solution: Divide each term by the coefficient of x^2:

$$x^2-\frac{3}{2}x-\frac{35}{2}=0.$$

Add $\frac{35}{2}$ to both sides of the equation:

$$x^2-\frac{3}{2}x=\frac{35}{2}.$$

Take one half of the coefficient of x, square it, and add it to both sides of the equation:

$$x^2-\frac{3}{2}x+\frac{9}{16}=\frac{35}{2}+\frac{9}{16}.$$

Change the left side into the form of the square of a binomial:

$$\left(x-\frac{3}{4}\right)^2=\frac{289}{16}.$$

Extract the square root:

$$x-\frac{3}{4}=\pm\frac{17}{4}.$$

Solve for x:

$$x_1=5$$

$$\text{and } x_2=-\frac{7}{2}.$$

Check by substituting $-\frac{7}{2}$ and 5 in the original equation.

Exercises

Solve the following quadratics by completing the square:

1. $a^2+4a-5=0.$
2. $x^2-2x=11.$
3. $8a=a^2-180.$
4. $2x+4=2+3x^2.$
5. $x^2-12x-13=0.$
6. $x^2-5x=12.$
7. $6x^2-51x+99=0.$
8. $a^2+3a=10.$
9. $4x^2+4x-35=0.$
10. $2a(a+4)=42.$
11. $2x^2-7x+3=0.$
12. $3a^2-a=2.$
13. $5x^2+2x=3.$
14. $4a^2-3a-10=0.$
15. $3x^2+5x-7=0.$
16. $6x^2-7x-5=0.$

136. A formula for solving quadratic equations. By solving the equation $ax^2+bx+c=0$, using the method explained in §135, it is possible to work out a formula which may be used to solve quadratic equations. The solution may be arranged as follows:

Divide each term by a: $x^2+\frac{b}{a}x+\frac{c}{a}=0.$

Subtract $\frac{c}{a}$ from both sides: $x^2+\frac{b}{a}x=-\frac{c}{a}$.

Add $\left(\frac{1b}{2a}\right)^2$ to both sides:

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}.$$

Change the left side into the square of a binomial: $$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}.$$

Carry out the subtraction on the right side: $$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}.$$

Extract the square root of both sides: $$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$=\pm\frac{\sqrt{b^2-4ac}}{2a}.$$

Subtract $\frac{b}{2a}$ from both sides: $$x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\text{or } x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

$$\text{and } x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Because these formulas are very important, they should be memorized.

137. How to use the quadratic formulas. To solve an equation by means of the quadratic formulas you proceed as follows:

1. Bring all terms to one side of the equation.
2. Collect similar terms.
3. Arrange the terms according to descending powers of x. The equation will then be of the form $ax^2+bx+c=0$.
4. By comparing the equation with $ax^2+bx+c=0$, determine the values of a, b, and c.

For example, in solving the equation $2x^2+5x-3=0$, comparison with $ax^2+bx+c=0$ determines the values $a=2$, $b=5$, $c=-3$.

5. Substitute these values in the formulas. Then

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{-5+\sqrt{25-4\cdot 2(-3)}}{2\cdot 2}$$

$$=\frac{-5+\sqrt{25+24}}{4}=\frac{-5+7}{4}=\tfrac{1}{2}$$

and $$x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{-5-\sqrt{49}}{4}=-\tfrac{12}{4}=-3.$$

Thus $x_1=\frac{1}{2}$

and $x_2=-3$.

Exercises

Solve the equations in Exercises 1 to 12 using the formulas, and check both results:

1. $x^2+8x+12=0$.

2. $a^2+5a+4=0$.

3. $x^2-10x+21=0$.

4. $3a^2-15a=-18$.

5. $3x^2-17x+10=0$.

6. $12x^2-x-1=0$.

7. $x^2+6x=16$.

8. $2x^2-3x-2=0$.

9. $18x-x^2=77$.

10. $2x^2-3x-9=0$.

11. $5x^2+3x=2$.

12. $3x^2-5x=8$.

Solve the following equations using the formulas:

13. $3x^2-2x-3=0$.

Solution: $x=\dfrac{2\pm\sqrt{4+36}}{6}=\dfrac{2\pm\sqrt{40}}{6}=\dfrac{2\pm2\sqrt{10}}{6}=\dfrac{1\pm\sqrt{10}}{3}$.

Therefore $x_1=\dfrac{1+\sqrt{10}}{3}$

and $x_2=\dfrac{1-\sqrt{10}}{3}$

are the exact values of the roots.

To find the approximate values extract the square root of 10. This gives 3.16.

$$\therefore x_1=\frac{1+3.16}{3}=\frac{4.16}{3}=1.39$$

$$\text{and } x_2=\frac{1-3.16}{3}=\frac{-2.16}{3}=-0.72.$$

14. $x^2-8x+14=0$.

15. $a^2-2a-4=0$.

16. $3x^2+8x=15$.

17. $3x^2-2x-3=0$.

18. $x^2-3x-5=0$.

19. $2a^2x^2+ax-3=0$.

20. $5x^2-12ax+4a^2=0$.

21. $3x^2-6ax+2a^2=0$.

22. $3c^2x^2-4cdx-4d^2=0$.

23. $10a^2x^2-4ax-3=0$.

Find the values of x to three places of decimals:

24. $x^2+5x=7$.

25. $3x^2-5x-8=0$.

26. $2x^2+5x-14=0$.

27. $x^2-3x-2=0$.

28. $\dfrac{a^2}{a-2}+\dfrac{4}{a-2}+5=0$.

29. $\dfrac{a-2}{a+2}=\dfrac{a+3}{a-3}-6\frac{2}{3}$.

Problems

138. Problems leading to quadratic equations. In the following problems choose your own method of solving the equations.

Exercises

1. The rate of the current of a stream is one mile an hour. It took us $7\frac{1}{2}$ hours to row downstream a distance of 18 miles and to return. What would be our rate of rowing in still water?

2. The constant difference between the successive terms of an arithmetical progression is -1. The first term is 5. How many terms are needed to make a sum equal to 5? Find two answers.

Suggestion: Use the formulas of §10.

3. A ball is thrown upward with a rate of 30 feet a second. How long will it take it to reach a height of 14 feet?

Suggestion: Use the formula $s=v_0 \cdot t-\frac{1}{2}gt^2$.

4. The total surface of a cone (Fig. 42) is given by the formula $T=\pi R(s+R)$, where R is the length of the radius of the base and s the slant height. Of what length must the radius be to make the total surface 374 square inches if the slant height is 10 inches?

Suggestion: Use $\pi=3\frac{1}{7}$.

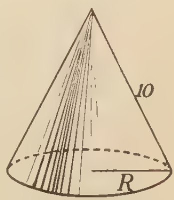


FIG. 42

5. A rectangular piece of tin is to be cut 4 inches longer than it is wide (Fig. 43). Four 6-inch squares are to be cut out, one from

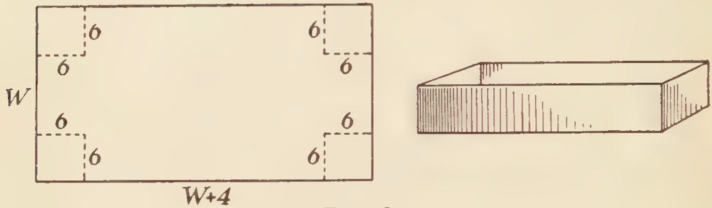


Fig. 43

each corner, so that when the sides are turned up an open box is formed which will contain 840 cubic inches. What should be the dimensions of the piece of tin?

6. An open box is to be made from a square piece of tin whose side is 16 inches. How large a square must be cut from each corner so that the box formed by turning up the sides contains 256 cubic inches?

Suggestion: Form the equation and use the method by factoring (§73) for finding one root. Find the other roots by solving the quadratic factor by means of the formula.

7. A coal bin is to be 8 feet deep and 10 feet longer than it is wide. If the bin is to hold 15 tons of coal, and if one ton (2000 pounds) occupies 22 cubic feet, what must be the length and width of the bin?

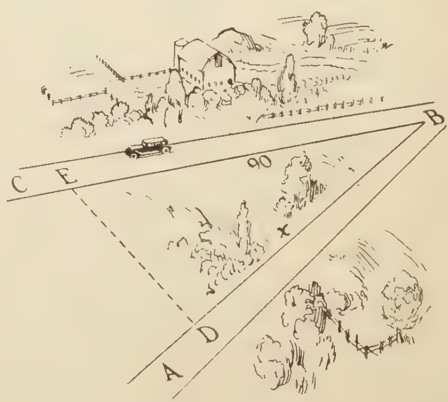


Fig. 44

8. A piece of land is to be laid off in the shape of a right triangle (Fig. 44) with the

vertex D of the right angle on AB. If the hypotenuse BE is to be 90 yards and if the perimeter is to be 216 yards, what must be the length of BD?

9. The radius of a circle is 21 inches long. By how much must it be decreased to diminish the area of the circle by 770 square inches?

Suggestion: Use $3\frac{1}{7}$ for π.

10. A rectangular garden bed is to be doubled in size by extending all sides as shown in Fig. 45. If the dimensions are 15 feet by 10 feet, by how much must the sides be increased?

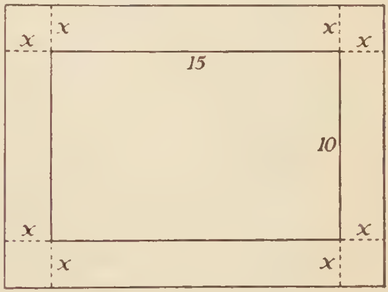


FIG. 45

11. Two men together can do a piece of work in $6\frac{2}{3}$ hours. If one alone can do it in 3 hours less than the other, how long will it take him to do it alone?

SUPPLEMENTARY TOPICS

139. The nature of the roots of a quadratic equation. You have learned that the roots of the general quadratic equation $ax^2+bx+c=0$ are

$x_1=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $x_2=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$. The number b^2-4ac which is found under the radical sign may be positive, zero, or negative, depending on the values of a, b, and c. These possibilities lead to the following classification of the roots:

1. **If b^2-4ac is positive,** the number $\sqrt{b^2-4ac}$ is said to be a *real* number.

Thus, for the equation $3x^2-7x+2=0$ the number $b^2-4ac=49-4\cdot3\cdot2=49-24=25$, *i.e.*, b^2-4ac is positive. The roots $\dfrac{7\pm\sqrt{25}}{6}$ are the two *real numbers* 2 and $\frac{1}{3}$.

For the equation $3x^2-7x+3=0$ the number $b^2-4ac=49-36=13$, *i.e.*, b^2-4ac is positive as before. But the exact values of the roots $\dfrac{7\pm\sqrt{13}}{6}$ cannot be found because 13 is not a square. They are said to be *irrational numbers.*

Hence, if b^2-4ac is a positive number, and

(a) if b^2-4ac is a *square*, the roots of the equation are *real* and *rational.*

(b) if b^2-4ac is not a square, the roots of the equation are *real* and *irrational.*

2. **If b^2-4ac is equal to zero,** then the number $\sqrt{b^2-4ac}$ is zero.

Thus, for the equation $x^2-6x+9=0$ the number $b^2-4ac=36-36=0$. The roots are $\dfrac{6+0}{2}$ and $\dfrac{6-0}{2}$, *i.e.*, the roots are both equal to 3.

Hence, if b^2-4ac is zero, the roots of the equation are *real, rational,* and *equal.*

3. **If b^2-4ac is less than zero,** then $\sqrt{b^2-4ac}$ denotes the square root of a negative number, and is called an *imaginary* number. Thus, for the equation $x^2-6x+10=0$ the number $b^2-4ac=36-40=-4$, and the roots are $\dfrac{6\pm\sqrt{-4}}{2}$. They are called *complex* numbers.

Briefly you may now summarize what you have learned about the roots of a quadratic equation:

(1) If $b^2-4ac<0$, the roots are complex.

(2) If b^2-4ac not <0, and if

- b^2-4ac is a perfect square, the roots are real, rational, and unequal.
- b^2-4ac is zero, the roots are real, rational, and equal.
- b^2-4ac is not a square, the roots are real, irrational, and unequal.

b^2-4ac is called the *discriminant* of the quadratic equation $ax^2+bx+c=0$.

140. The geometric interpretation of the nature of the roots of a quadratic equation. Let equation (1): $x^2-6x+5=0$, be represented graphically (§17) as shown in Fig. 46. From the graph the roots of the

Carl Friedrich Gauss (1777–1855) was the son of a bricklayer. He showed early great mathematical ability. His favorite study was higher arithmetic, but his contributions were made in almost every field of mathematics. In time he became the greatest mathematician of Germany and the founder of modern mathematics of Germany. He was the first to use the term "complex number" in the sense it has today. Formerly, imaginary numbers appearing as roots of equations had been regarded as meaningless. Through his work the imaginary number was fully admitted into the number system.

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equation are shown to be $x_1 = 1$, $x_2 = 5$, *i.e.*, the roots are real, rational, and unequal.

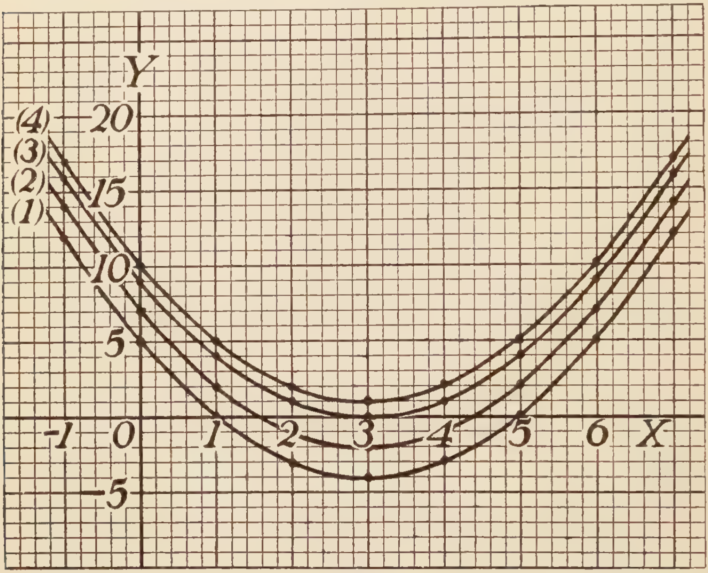


FIG. 46

The discriminant $b^2 - 4ac = 36 - 20 = 16$, which is a perfect square. This verifies what was said in the summary (§139).

For equation (2): $x^2 - 6x + 7 = 0$, the number $b^2 - 4ac = 36 - 28 = 8$. Hence, the roots are real, irrational, and unequal. In the diagram, the roots are the point of intersection of the x-axis with curve (2).

For equation (3): $x^2 - 6x + 9 = 0$, you have $b^2 - 4ac = 36 - 36 = 0$. The roots are real, rational, and equal. Geometrically this means that curve (3) just touches the x-axis.

For equation (4): $x^2-6x+10=0$, we have $b^2-4ac=36-40=-4$. The roots are therefore complex. Geometrically this means that the curve lies entirely above the x-axis, *i.e.*, it does not touch the axis and it does not intersect it, as shown in curve (4).

Exercises

Determine the nature of the roots of the following equations:

1. $3x^2+8x+5=0$.

Solution: $b^2-4ac=64-60=4$.

Since 4 is a square, the roots are real, rational, and unequal.

2. $x^2-8x+16=0$.

Solution: $b^2-4ac=64-64=0$.

$\therefore$ The roots are real, rational, and equal.

3. $x^2-5x+8=0$.

Solution: $b^2-4ac=25-32=-7$. Hence the roots are complex.

4. $5x^2-3x-2=0$.
5. $2x^2-4x+1=0$.
6. $x^2-2x+5=0$.
7. $7x^2+9x-10=0$.
8. $3x^2-x+10=0$.
9. $3x^2-7x+2=0$.
10. $2x^2-13x+15=0$.
11. $x^2-3x+5=0$.
12. $9x^2+12x+4=0$.
13. $5x^2+8x-2=0$.
14. $5x^2-4x+2=0$.
15. $x^2-5x+8=0$.

141. Relations between the roots of a quadratic equation and the coefficients. You know that the roots of the equation $ax^2+bx+c=0$ are

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

$$\text{and } x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Find the *sum* of the two roots as follows:

$$x_1+x_2=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-2b}{2a}=-\frac{b}{a}.$$

$$\therefore x_1+x_2=-\frac{b}{a}.$$

Find the *product* of the two roots as follows:

$$x_1\cdot x_2=\frac{\left(-b+\sqrt{b^2-4ac}\right)\left(-b-\sqrt{b^2-4ac}\right)}{(2a)(2a)}$$

$$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}$$

$$=\frac{b^2-b^2+4ac}{4a^2}$$

$$=\frac{\cancel{4}\cancel{a}c}{\cancel{4}\cancel{a^2}_{a}}=\frac{c}{a}.$$

$$\therefore x_1\cdot x_2=\frac{c}{a}.$$

This means that if every term of $ax^2+bx+c=0$ is divided by a, and the equation is thereby changed to $x^2+\frac{b}{a}x+\frac{c}{a}=0$, then the coefficient of x with the sign changed is the sum of the roots, and the constant term is the product of the roots.

Exercises

State the sum of the roots and the product of the roots for each of the following equations:

1. $x^2-3x+7=0$.

Solution: $x_1+x_2=-\frac{b}{a}=-\frac{-3}{1}=3$

$$x_1 \cdot x_2=\frac{c}{a}=\frac{7}{1}=7.$$

2. $3x^2-2x+8=0$.

Solution: $x_1+x_2=-\frac{-2}{3}=\frac{2}{3}$.

$$x_1 \cdot x_2=\frac{8}{3}.$$

3. $2x^2-9x+8=0$.

4. $3x^2-2x+6=0$.

5. $5x^2-2x-16=0$.

6. $7x^2+9x-10=0$.

7. $5x^2-3x-2=0$.

8. $x^2-x+10=0$.

9. $x^2+12x+35=0$.

10. $x^2-6x+9=0$.

Form the equations whose roots are:

11. 2, 5.

Solution: $x_1=2$, $x_2=5$

$$\therefore -(x_1+x_2)=-7=-\frac{b}{a}$$

$$x_1 \cdot x_2=10=\frac{c}{a}.$$

Substituting in the equation $x^2+\frac{a}{b}x+\frac{c}{a}=0$, you have

$$x^2-7x+10=0.$$

12. $-2, 3$.

13. $-5, -2$.

14. $\frac{1}{2}, 3$.

15. $\frac{3}{4}, \frac{1}{2}$.

16. $-\frac{3}{4}, -\frac{1}{3}$.

17. $\frac{1}{2}, -\frac{1}{4}$.

18. a, b.

19. $a+2b, a$.

20. $-m+n, -m-n$.

21. $a-2, a+2$.

22. $\sqrt{3}, -\sqrt{3}$.

23. $1+\sqrt{2}, 1-\sqrt{2}$.

24. $2+\sqrt{3}, 2-\sqrt{3}$.

25. $-1+\sqrt{7}, -1-\sqrt{7}$.

26. $a+\sqrt{b}, a-\sqrt{b}$.

142. Equations that are of quadratic form. Some equations are similar to quadratic equations and can

therefore be solved by the same methods. For example, the equation $(4a+5)^2+2(4a+5)-15=0$ when compared with $ax^2+bx+c=0$ shows that you may put $4a+5$ equal to x and let $a=1$, $b=2$, $c=-15$.

The solution of $(4a+5)^2+2(4a+5)-15=0$ is the same as that of $x^2+2x-15=0$. The complete solutions of the two equations are as follows:

Solutions:

$x^2+2x-15=0.$	$(4a+5)^2+2(4a+5)-15=0.$
Factor:	
$(x-3)(x+5)=0.$	$(4a+5-3)(4a+5+5)=0.$
$\therefore x-3=0$	$\therefore 4a+2=0$
and $x+5=0.$	and $4a+10=0.$
$\therefore x_1=3$	$\therefore a_1=-\frac{1}{2}$
and $x_2=-5.$	and $a_2=-2\frac{1}{2}.$

EXERCISES

Solve the following equations:

1. $(x-2)^2-7(x-2)+12=0.$
2. $(x+3)^2-2(x+3)-3=0.$
3. $x^2+4x+4+11(x+2)+30=0.$
4. $(3x-5)^2-8(3x-5)+7=0.$
5. $x^2-18x+81+9(x-9)+20=0.$
6. $a^4-26a^2+25=0.$

Solution: Change $a^4-26a^2+25=0$ to $(a^2)^2-26(a^2)+25=0.$

Factor: $(a^2-25)(a^2-1)=0.$

$$\therefore a^2-25=0$$
$$\text{and } a^2-1=0.$$
$$\therefore a^2=25$$
$$\text{and } a^2=1.$$
$$\therefore a_1=5.$$
$$a_2=-5.$$
$$a_3=1.$$
$$a_4=-1.$$

7. $6y^4-13y^2+6=0.$
8. $36a^4-97a^2+36=0.$
9. $x^4-20x^2+99=0.$
10. $x-8x^{\frac{1}{2}}+15=0.$
11. $a-13a^{\frac{1}{2}}+40=0.$
12. $y-6y^{\frac{1}{2}}+8=0.$

143. Irrational equations leading to quadratics. Irrational equations are sometimes of quadratic form. For example the equation

$x^2-3x+15+\sqrt{x^2-3x+15}-30=0$ may be written

$$(\sqrt{x^2-3x+15})^2+\sqrt{x^2-3x+15}-30=0.$$

Factoring, you have

$$(\sqrt{x^2-3x+15}+6)(\sqrt{x^2-3x+15}-5)=0.$$

$$\therefore \sqrt{x^2-3x+15}+6=0$$

$$\text{and } \sqrt{x^2-3x+15}-5=0.$$

$$\therefore \sqrt{x^2-3x+15}=-6$$

$$\text{and } \sqrt{x^2-3x+15}=5.$$

Squaring both members in the last two equations, you have $$x^2-3x+15=36$$

$$\text{and } x^2-3x+15=25.$$

These equations are quadratic equations.

Solve them and substitute the roots in the original equation to see whether they check. The check is necessary for irrational equations because in squaring a radical equation a new root is sometimes introduced which is not a root of the original equation.

Exercises

Solve the following:

1. $x^2-5x-2+2\sqrt{x^2-5x-2}=8.$

2. $3(2a^2-a)-\sqrt{2a^2-a}-2=0.$

3. $a-9+9\sqrt{a-9}+20=0.$

4. $a^2-8a+40-2\sqrt{a^2-8a+40}=35.$

5. $2(x^2-x+1)+3\sqrt{x^2-x+1}=5.$

6. $y+2\sqrt{y-1}-4=0.$

Solution: Subtract $y-4$. This leaves the radical alone on one side, and $2\sqrt{y-1}=4-y$.

Square both sides:

$$4(y-1)=16-8y+y^2.$$

$$\therefore y^2-12y+20=0.$$

$$\therefore y_1=10$$

$$\text{and } y_2=2.$$

Check by substituting in the original equation.

7. $a+\sqrt{a-7}=19.$

8. $\sqrt{x+9}-\sqrt{x-7}=2.$

9. $\sqrt{a+5}-\sqrt{a}=1.$

10. $\sqrt{x^2-16}-\sqrt{x^2-9}=7.$

11. $2\sqrt{a-1}=\sqrt{4a+3}-1.$

12. $\sqrt{4a-22}+\sqrt{2}-2\sqrt{a}=0.$

13. $\sqrt{x}+\sqrt{4x+9}-\sqrt{x+5}=0.$

14. $\sqrt{2+a}=\sqrt{3}-\sqrt{2-a}.$

15. $\sqrt{2a+8}=\sqrt{7a+21}-\sqrt{a+5}.$

16. $\sqrt{x+1}+\sqrt{x+10}=9.$

17. $\sqrt{2x+5}+\sqrt{2x-3}=4.$

18. $2\sqrt{a-1}-\sqrt{a-4}=\sqrt{a+4}.$

19. $x^2-6x+5\sqrt{x^2-6x+20}=46.$

Suggestion: Add 20 to both sides of the equation.

20. $3a^2+a+5\sqrt{3a^2+a+6}=30.$

144. What every pupil should be able to do. Chapter X completes the study of quadratic equations. You should now be able to do the following:

1. To solve quadratic equations
   a. by factoring,
   b. by completing the square,
   c. by formula.
2. To solve problems leading to quadratic equations.
3. To interpret the solution of a quadratic equation by means of the graph.

Those who have studied the supplementary topics should be able:

4. To form an equation whose roots are given.
5. To determine the nature of the roots of a quadratic equation.
6. To solve equations of quadratic form.
7. To solve irrational equations.

145. Typical problems and exercises. The following problems and exercises are typical of the chapter:

1. Solve for x: $6x^2-200=0$.

2. Solve by factoring: $3a^2-15a=0$.

3. Solve by various methods: $2x^2-7x+3=0$.

4. Study again the exercises of §138.

The following are typical of the supplementary work:

5. Determine the nature of the roots of the following:

$5a^2-3a-2=0$. $\qquad 3m^2-m+10=0$.

$x^2+6x+9=0$. $\qquad x^2-2x-5=0$.

6. Form the equations whose roots are $\frac{1}{2}$, $-\frac{3}{4}$; $1+\sqrt{2}$, $1-\sqrt{2}$.

7. Solve: $(x+3)^2-2(x+3)-3=0$.

8. Solve: $a^4-20a^2+99=0$.

9. Solve: $\sqrt{x+9}-\sqrt{x-7}=2$.

CHAPTER XI

LINEAR AND QUADRATIC EQUATIONS WITH TWO UNKNOWNS

Graphic Solution

146. What you are going to study in Chapter XI. Consider the following problem: It is required to draw two circles (Fig. 47), just touching each other, so that the distance AA' between the centers, *i.e.*, the sum of the radii, is 4 feet, and that the sum of the areas is $81\frac{5}{7}$ square feet.

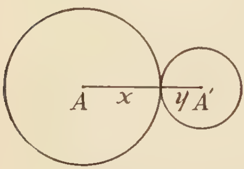


Fig. 47

To solve the problem denote the radii by x and y respectively.

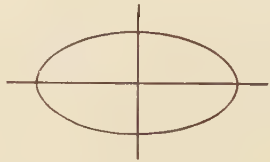
Then show that $\pi x^2 + \pi y^2 = 81\frac{5}{7}$
and that $x + y = 4$.

The first of these equations is of the second degree and contains two unknowns. The second equation is linear. In this chapter you are going to study ways of solving systems of equations with two unknowns when one of the equations is linear and the other quadratic.

147. How to represent graphically a quadratic equation with two unknowns. In a more advanced course in mathematics it is shown that the graph of a

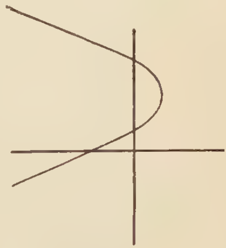
quadratic equation with two unknowns is one of the following cases: a *circle** (Fig. 48), an *ellipse* (Fig. 49),

Circle. FIG. 48

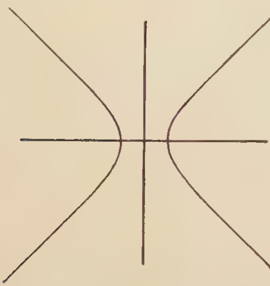


Ellipse. FIG. 49

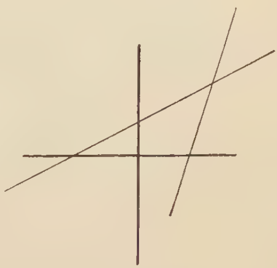
a *parabola* (Fig. 50), a *hyperbola* (Fig. 51), or a pair of straight lines (Fig. 52). The form of the equation determines which of these lines will be obtained by making the graph. Only the most important forms will be studied in this chapter.



Parabola. FIG. 50



Hyperbola. FIG. 51



Two lines. FIG. 52

*The Greeks made a study of the parabola, ellipse, and hyperbola. They were able to show that the curves were obtained by cutting a cone. Johann Kepler (1571–1630) showed that the earth revolves around the sun in an ellipse. Comets travel in orbits of the shape of parabolas and ellipses, and an object thrown in any but a vertical direction describes a parabola as the path.

The following are illustrations of the types of equations which are to be represented graphically:

(a) $y=2x^2-5x-8$.
(b) $x^2+y^2=16$.
(c) $3x^2+2y^2=36$.
(d) $3x^2-4y^2=48$.
(e) $xy=12$.
(f) $(2x+y-3)(x-y+5)=0$.

In previous courses you have become familiar with the method of drawing graphs. However, the following suggestions will be helpful in making graphs of the quadratic equations above:

1. To save time, solve the equation for one of the two unknowns. Thus, for the equation $x^2+y^2=16$ you will have $y=\pm\sqrt{16-x^2}$, or $x=\pm\sqrt{16-y^2}$.

2. In the right side substitute values for the unknown and then compute the corresponding values of the other unknown. For example let $y=0, 1, 2, \ldots$ Then $x=\pm 4, \pm\sqrt{15}, \pm\sqrt{12}, \ldots$

3. Tabulate the pairs of corresponding values.

4. Draw the reference axes and select convenient units.

5. Plot the number pairs found in the table, and join the points by a smooth curve.

148. To solve graphically a system of equations with two unknowns if one equation is linear and the other quadratic. Let it be required to solve the system

$$x=y^2-6y+8 \qquad (1)$$
$$x-y-2=0. \qquad (2)$$

Equation (1) is a quadratic, and equation (2) is linear.

To solve the system graphically make the graph of each equation (Fig. 53). The graph of (1) is a curved line, which is called *parabola*. The second is a straight line.

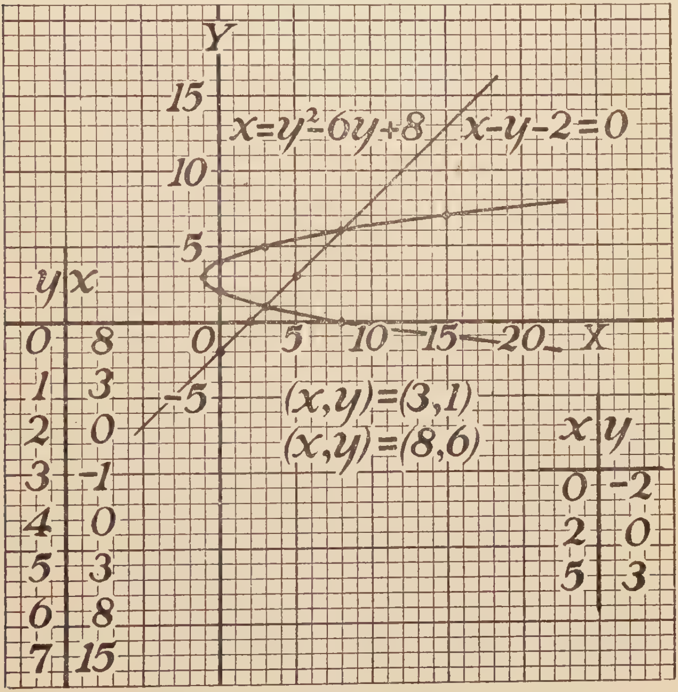


FIG. 53

Find the two points of intersection of the parabola with the straight line.

For each point determine the pair of corresponding values of x and y.

Each pair is a solution of the system.

Thus, you have two solutions: $(x, y) = (8, 6)$

and $(x, y) = (3, 1)$.

Check by substituting each pair in the original equations.

In general, a system of equations, one of which is linear and one quadratic, has two solutions. If the straight line just touches the curve (Fig. 54) the graph shows that there is only one solution.

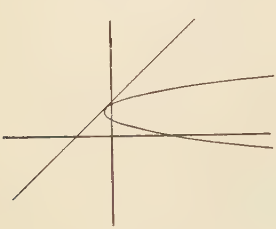


FIG. 54

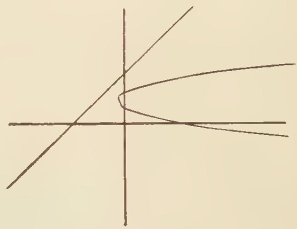


FIG. 55

If the straight line does not intersect the curve (Fig. 55), the solution cannot be found graphically.

EXERCISES

Solve the following systems graphically:

1. $y^2+3y-x=18$
   $x-2y=2.$

2. $x^2-5x-y+6=0$
   $x-y+1=0.$

3. $y^2+4x-17=0$
   $2x-y=1.$

4. $y^2-x-6=0$
   $x-y=0.$

5. $x^2-5x+3y=6$
   $2x-3y=4.$

6. $x^2-y=0$
   $-4x+5y=28.$

7. $y^2+y+7=x$
   $x-4y-5=0.$

8. $y^2+3x-27=0$
   $x-y+3=0.$

9. $-x^2-4x+69=3y$
   $x=2y.$

10. $3y^2-9y-x-2=0$
    $3y-x-2=0.$

149. How to recognize the equation of a parabola. The graphs of all the quadratic equations of §148 are

parabolas. Any equation of the form $y=ax^2+bx+c$ or $x=ay^2+by+c$ is a parabola.

150. Graphical solution of a system of equations one of which represents a circle. The graph of the equation $x^2+y^2=25$ is a *circle.* To draw the graph solve the equation for x or y. This gives $y=\pm\sqrt{25-x^2}$.

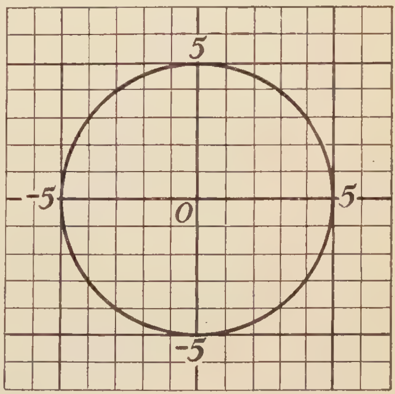
Let $x=0, 1, 2, 3, 4, 5$.

Then $y=\pm 5, \pm\sqrt{24}, \pm\sqrt{21}, \pm 4, \pm 3, 0$
$=5, \pm 4.8, \pm 4.5, \pm 4, \pm 3, 0$.

Tabulate the pairs of values of x and y (Fig. 56).

Plot the corresponding points, and draw the curve.

Any equation of the form $x^2+y^2=a^2$ represents a circle, whose radius is a. Hence, you may draw the graph of $x^2+y^2=25$ with the compass by making a circle whose radius is 5.



x	0	1	2	3	4	5
y	± 5	± 4.8	± 4.5	± 4	± 3	0

FIG. 56

EXERCISES

Solve the following systems graphically:

1. $x^2+y^2=25$
$x-y=-1$.

2. $x^2+y^2=20$
$x+y=6$.

3. $x^2+y^2=25$
$x+y=1$.

4. $x^2+y^2=100$
$x-y+2=0$.

5. $x^2+y^2=5$
$x-y=1.$

6. $x^2+y^2=36$
$5x+y=23.$

7. $x^2+y^2=25$
$y-3x=5.$

8. $x^2+y^2=12$
$y=4-2x.$

151. Graphical solution of a system of equations one of which is an ellipse. The graph of the equation $16x^2+25y^2=400$ is an *ellipse* (Fig. 57). It is made as follows:

1. Solve the equation for y. This gives $y=\pm\frac{4}{5}\sqrt{25-x^2}$.

2. Let $x=0, 1, 2, 3, 4, 5$. Show that $y=\pm 4, \pm 3.9, \pm 3.7, \pm 3.2, \pm 2.4, 0$.

3. Plot the corresponding points and draw a smooth curved line through them. This is the required ellipse.

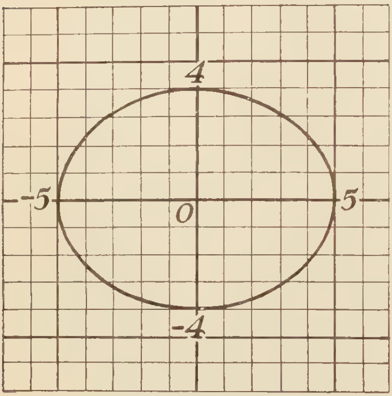


Fig. 57

The equation of the ellipse (Fig. 57) resembles the equation of the circle (Fig 56). Both have two second-degree terms, one containing x^2 and the other y^2. However, the coefficients of x^2 and y^2 are the same for the circle and are different for the ellipse. The signs of the coefficients are the same.

Exercises

Solve the following systems graphically using the method shown in §148:

1. $x^2+4y^2=25$
$x=2y+2.$

2. $4x^2+9y^2=36$
$2x-3y=0.$

3. $3x^2+5y^2=23$
$2x - y = 0.$

4. $x^2+4y^2=32$
$5x+6y=8.$

5. $x^2+9y^2=81$
$x=3y+5.$

6. $4x^2+y^2=36$
$2x-y=1.$

7. $36x^2+y^2=288$
$y=6x-24.$

8. $x^2+4y^2=48$
$x=3+2y.$

152. Graphical solution of a system of equations one of which is a hyperbola. Comparing the equation $4x^2-9y^2=36$ with that of the ellipse $4x^2+9y^2=36$ studied in §151, you will note that the equations are of the same form, except for the difference in the signs of the coefficients of x^2 and y^2.

The equation $4x^2-9y^2=36$ is a *hyperbola* (Fig. 58.) The method of making the graph is the same as the method explained in §151.

Verify the correctness of the number pairs in the table of Fig. 58. Note that there are no real values of y corresponding to $x=1$, 2, and 3.

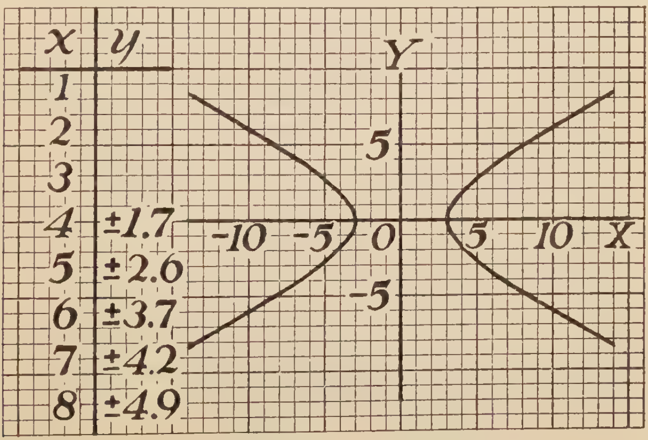


Fig. 58

Exercises

Solve the following systems of equations graphically:

1. $x^2-y^2=9$
   $x+y=9.$

2. $4x^2-y^2=16$
   $2x+y=8.$

3. $x^2-y^2=27$
   $x=3+y.$

4. $3x^2-y^2=3$
   $x+2y=4.$

5. $x^2-2y^2=8$
   $x=3+2y.$

6. $3x^2-7y^2=5$
   $2x+y=5.$

Algebraic Solution

153. A linear and a quadratic equation have always two solutions. The graphical method has shown that in general a system containing one linear and one quadratic equation has two solutions, and that the solutions are the number pairs corresponding to the two points of intersection of the curve and the straight line which represent the equations.

When the straight line does not intersect the curve, the graphical method fails to find the solutions. The algebraic method, however, enables us to find the two solutions in every case. The method is explained in the following examples:

1. Solve the system: $y=x^2-4x+1 \quad (1)$
   $y=1-2x. \quad (2)$

Solution: Substitute $1-2x$ for y in equation (1).
This gives $1-2x=x^2-4x+1.$

$$\therefore x^2-2x=0.$$
$$x(x-2)=0.$$
$$\therefore x_1=0.$$
$$x_2=2.$$

Substitute these values in equation (2):

$$y_1 = 1.$$
$$y_2 = 1 - 4 = -3.$$

$\therefore$ The solutions are $(x_1, y_1) = (0, 1)$
and $(x_2, y_2) = (2, -3)$.

2. Solve the system: $4x^2 + 9y^2 = 36$
$x - 3y = 2.$

Solution: (1) Solve the *linear* equation for one of the variables. Thus, $x = 2 + 3y$.

(2) Substitute $2+3y$ for x in the *quadratic* equation. Then $4(2+3y)^2 + 9y^2 = 36$
or $4(4+12y+9y^2) + 9y^2 = 36$.

Arrange according to powers of y and solve:

$$45y^2 + 48y - 20 = 0.$$

$$\therefore y = \frac{-48 \pm \sqrt{48^2 + 4 \cdot 45 \cdot 20}}{2 \cdot 45}$$

$$= \frac{-48 \pm 12\sqrt{41}}{90}$$

$$= \frac{-8 \pm 2\sqrt{41}}{15}.$$

$$\therefore x = 2 + 3y = 2 + \frac{-8 \pm 2\sqrt{41}}{5}$$

$$= \frac{2 \pm 2\sqrt{41}}{5}.$$

$$\therefore (x_1, y_1) = \left(\frac{2+2\sqrt{41}}{5}, \frac{-8+2\sqrt{41}}{15}\right).$$

$$(x_2, y_2) = \left(\frac{2-2\sqrt{41}}{5}, \frac{-8-2\sqrt{41}}{15}\right).$$

Exercises

Solve the following systems algebraically by solving the linear equation and then substituting in the quadratic:

1. $s=\frac{1}{2}gt^2$
   $s=3t-5.$

2. $y=3x^2-9x-2$
   $3x-y=2.$

3. $x=5+2y-y^2$
   $x-2y=3.$

4. $x^2+y^2=25$
   $x-y=7.$

5. $x^2-xy+y^2=7$
   $x+y=4.$

6. $x^2+y^2+xy=28$
   $x-y=2.$

7. $x^2+y^2=25$
   $y-x+1=0.$

8. $x^2+y^2=20$
   $x+y=6.$

9. $x^2+y^2=25$
   $x+y=5.$

10. $xy=12$
    $x-y+1=0.$

11. $x^2+y^2=25$
    $3x-4y=0.$

12. $xy=2$
    $x+y=3.$

13. $x^2+xy+y^2=7$
    $2x+3y=0.$

14. $x^2-xy+y^2=3$
    $x+y=3.$

15. $2x^2-xy+y^2=16$
    $y-2x=0.$

16. $xy=6$
    $x+y=5.$

17. $x^2+3y^2=28$
    $3x+y=0.$

18. $2x^2+y^2=15$
    $x-y=3.$

Problems

154. Problems leading to systems containing one linear and one quadratic equation. The following problems should be worked with two unknown literal numbers. Derive the equations and solve. Any solution of the equations which is not a solution of the problem should be discarded.

Exercises

1. Two men A and B start from the same point. One walks east, the other west. They find that after 5 hours the first has walked 5 miles farther than the second and that they are 25 miles apart. How fast did each walk?

2. In an arithmetical progression the nth term is given by the formula $l=a+(n-1)d$ and the sum of n terms by the formula $s=\frac{n}{2}(a+l)$. Find l and n when the first term is $\frac{1}{3}$, the common difference is $-\frac{1}{12}$, and the sum of n terms is $-\frac{3}{2}$.

3. It is required to lay off a field in the form of a right triangle using as hypotenuse a base line 10 rods long. What must be the lengths of the other two sides of the triangle if 23 rods of wire fencing are to be used to inclose the field?

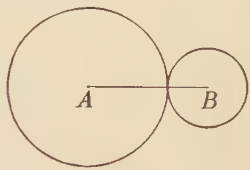


Fig. 59

4. Two circles (Fig. 59) are to be constructed with A and B as centers so that the area of the larger is $392\frac{1}{2}$ square inches greater than that of the smaller. If the distance between A and B is 25 inches, what are the lengths of the radii?

5. On the sides of a right angle ACB (Fig. 60) lengths equal to 3 and 4 inches are marked off. A and B are joined by a straight line. It is required to draw a line through C dividing ACB into two triangles having equal perimeters. Locate the point on AB to which the line should be drawn.

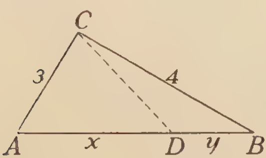


Fig. 60

6. The area of a right triangle is 24 square yards, the perimeter is 24 yards, and the hypotenuse is 10 yards. Find the sides including the right angle

7. The sum of a given number of terms in an arithmetical progression is 121. The last term is 21, and the common difference is 2.

Using the formulas $l=a+(n-1)\,d$ and $s=\frac{n}{2}(a+l)$ find the first term and the number of terms.

8. The first term of an arithmetical progression is 3, the sum of n terms is 300, and the common difference is 4. Find n and the nth term.

9. Solve the problem stated in §146.

Derive the equations for the following exercises. You need not solve the equations.

10. One of two numbers exceeds the other by 9, and the sum of their squares is 221. Find the numbers.

11. By increasing each side of a rectangle by 2 feet the area is increased by 38 square feet. If the diagonal is 13 feet long, what are the dimensions?

12. One number exceeds another by 2, and the square of the sum is 16 greater than 6 times the sum. What are the numbers?

13. The difference of the areas of two squares is 252 square feet, and the difference between their sides is 6 feet. How long are the sides?

14. The sum of two adjacent sides of a parallelogram is 12 inches. The sum of the squares is 80 square inches. How long is each?

155. What everybody should be able to do. Chapter XI has taught the following:

1. To make the graph of a quadratic equation containing two unknowns.

2. To solve a system of equations one of which is linear and one quadratic

a. by means of graphs,

b. by elimination, *i.e.*, by solving the linear equation for one of the unknowns and substituting in the quadratic equation.

3. To solve problems leading to equations in two unknowns.

156. Typical problems and exercises. The following illustrate types of problems and exercises you should be able to solve. Solve the equations both graphically and algebraically.

1. $x^2+2x-y=7$
$x-2y=0.$

2. $4x^2+9y^2=100$
$2x+9y=-10.$

3. $xy=6$
$x+y=5.$

4. Solve the following problem: The perimeter of a right triangle is 36 inches, and the area is 54 square inches. If the hypotenuse is 15 inches find the lengths of the other two sides.

CHAPTER XII

SUPPLEMENTARY TOPICS*

INFINITE GEOMETRIC PROGRESSIONS

157. Examples of infinite geometric progressions. If in the geometric progression $2+4+8+16+\ldots$ the number of terms continues to increase without bound, it is called an *infinite progression.* Similarly, the progression $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ is an infinite progression, if the number of terms increases indefinitely.

In the geometric progression $2+4+8+\ldots$ the common ratio is 2. Whenever the ratio is positive and *greater* than 1 each term is greater than the preceding term and the progression is said to be *increasing.* Give examples of other increasing progressions.

If the common ratio is positive and *less* than 1 the progression is *decreasing,* each term being less than the preceding one, as in the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ Give other examples of decreasing geometric progressions.

The recurring decimals of arithmetic are illustrations of infinite geometric progressions. Thus, .333 . . . may be written $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\ldots$, which is a geometric progression having $a=\frac{3}{10}$ and $r=\frac{1}{10}$.

*Chapter XII is intended for advanced pupils who wish to do work in addition to the requirements of the course.

158. The sum of an infinite geometric progression. If in a progression the common ratio r is greater than 1, the sum s of n terms increases as n increases. Thus, in the progression $3+6+12+24+ \ldots$

$$s_1=3.$$
$$s_2=3+6=9.$$
$$s_3=3+6+12=21.$$
$$s_n=3+6+12+ \ldots \text{ to } n \text{ terms.}$$

If n increases without limit, the sum also is unlimited.

A geometric progression may have a limited sum although the number of terms is unlimited. For example, the progression $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+ \ldots$ has a definite limited sum, as the number of terms increases without bound, as may be seen geometrically (Fig. 61).

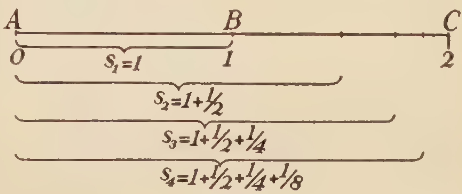


FIG. 61

As shown in the diagram, mark off segments AB and BC equal to 1. This gives $s_1=1$, leaving a remainder $BC=1$.

Then add to AB a segment equal to $\frac{1}{2}$ of BC. This gives $s_2=1+\frac{1}{2}$.

To s_2 add $\frac{1}{2}$ of the remainder. which gives

$$s_3=1+\frac{1}{2}+\frac{1}{4}.$$

Since there is always one half of the remainder left over, the process of laying off segments may be continued indefinitely. As the sum of the segments increases, approaching nearer and nearer the length 2,

it can be made to differ from the segment whose length is 2 by less than any length, however small. The *sum of the segments* is said to be a length equal to 2.

Similarly, the arithmetical sum $s_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ increases as n increases, and approaches 2 as n increases without bound.

The various partial sums are tabulated below:

n	1	2	3	4	5	6	7	8	9
s_n	1	1.5	1.75	1.875	1.9375	1.96875	1.984375	1.9921875	1.99609375

The table shows the changes of the value of s_n as n changes from 1 to 9.

The changes of $s_n = 1 + \frac{1}{2} + \frac{1}{4} + \ldots$ have been shown in two different ways: (1) Geometrically, as in Fig. 61, and (2) arithmetically, as in the table above. The same changes may be shown algebraically by using the formula for finding the sum of n terms of a geometric progression.

The formula is $s_n = \dfrac{a - ar^n}{1 - r}$.

Let $a = 1$ and $r = \frac{1}{2}$.

$$\text{Then } s_n = \frac{1 - 1(\frac{1}{2})^n}{1 - \frac{1}{2}} = \frac{1 - (\frac{1}{2})^n}{\frac{1}{2}} = \frac{1}{\frac{1}{2}} - \frac{(\frac{1}{2})^n}{\frac{1}{2}}$$
$$= 2 - 2(\tfrac{1}{2})^n.$$

As n increases, $(\frac{1}{2})^n$ decreases. The table below contains corresponding values of n and $(\frac{1}{2})^n$.

n	1	2	3	4	5	6	7	8
$(\frac{1}{2})^n$ as a fraction	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
$(\frac{1}{2})^n$ as a decimal	.5	.25	.125	.0625	.03125	.015625	.0078125	.00390625

As n increases without bound, $(\frac{1}{2})^n$ approaches the value zero. Hence, $s_n = 2 - 2(\frac{1}{2})^n$ approaches $2 - 2 \cdot 0$, or 2.

The progression $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$, which has been shown to have a limited sum although an unlimited number of terms are used, is but a special case of the general case of a geometric progression in which the common ratio r is numerically less than 1. The formula for finding the sum for the general case is derived as follows: Let the sum of n terms be represented by the formula

$$s_n = \frac{a - ar^n}{1 - r}.$$

This may be changed to

$$s_n = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

If r is numerically less than 1 and if n increases indefinitely, then $(r)^n$ approaches zero.

Therefore ar^n approaches zero,

and $\frac{ar^n}{1-r}$ approaches zero.

It follows that

$$s = \frac{a}{1-r},$$

i.e., *the sum to infinity of a geometric progression in which r is numerically less than 1 is found by means of the formula* $\boldsymbol{s = \frac{a}{1-r}}$.

Exercises

Find the sums to infinity of the progressions in Exercises 1 to 13:

1. $3+1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$

Solution: Let $a=3$ and $r=\frac{1}{3}$.

Since r is less than 1, and since the progression has an unlimited number of terms, we may use the formula $s=\dfrac{a}{1-r}$.

Thus $s=\dfrac{3}{1-\frac{1}{3}}=3\times\frac{3}{2}=\frac{9}{2}$, or $4\frac{1}{2}$.

2. $12+3+\frac{3}{4}+\ldots$

3. $3+\frac{1}{2}+\frac{1}{12}+\frac{1}{72}+\ldots$

4. $10+5+2\frac{1}{2}+1\frac{1}{4}+\ldots$

5. $\frac{4}{3}+\frac{2}{3}+\frac{1}{3}+\ldots$

6. $2+\frac{2}{3}+\frac{2}{9}+\ldots$

7. $16+4+1+\frac{1}{4}+\ldots$

8. $1-\frac{1}{5}+\frac{1}{25}-\ldots$

Suggestion: Let $a=1$ and $r=-\frac{1}{5}$.

9. $1-\frac{1}{3}+\frac{1}{9}-\ldots$

10. $6-4+\frac{8}{3}-\ldots$

11. $-2-\frac{1}{4}-\frac{1}{32}-\ldots$

12. $3-6+12-24+\ldots$

13. $32-16+8-4+\ldots$

Find the values of the recurring decimals in Exercises 14 to 26:

14. $.2727\ldots$

Solution: Change $.272727\ldots$ to $\dfrac{27}{100}+\dfrac{27}{100^2}+\dfrac{27}{100^3}+\ldots$

$$\text{Let } a=\frac{27}{100},\ r=\frac{1}{100}.$$

$$\text{Then } s=\frac{a}{1-r}=\frac{\frac{27}{100}}{\frac{99}{100}}=\frac{27}{99}=\frac{3}{11}.$$

15. .6666 . . .

16. .3030 . . .

17. .3939 . . .

18. .3636 . . .

19. .481481 . . .

20. .234234 . . .

21. .520520 . . .

22. .83333 . . .

23. .16666 . . .

24. .0171717 . . .

25. 3.161616 . . .

26. 25.363636 . . .

Solve the following problems:

27. A ball falls to the ground from a height of 15 feet, bounces back approximately to $\frac{1}{5}$ of the height, falls again to the ground and bounces back approximately $\frac{1}{5}$ of the height which it fell and continues to fall and bounce back. Find the total distance traveled by the ball.

28. A line segment is 4 inches long. If you mark off $\frac{1}{3}$ of the segment, and then $\frac{1}{3}$ of the remainder, etc., what is the sum of all the parts marked off?

29. The sum to infinity of a geometric progression is 64 times the sum to 6 terms. Find the common ratio.

30. The first and fourth terms of a geometric progression are 225 and $14\frac{2}{5}$. State the progression and find the sum to infinity.

Inconsistent and Equivalent Equations

159. What is meant by inconsistent equations. The equations $x+y=6$ and $x+y=8$ cannot both be satisfied by the same pair of values of x and y. They are said to be *contradictory* or *inconsistent* equations. It is usually possible to determine by inspection when two equations are inconsistent if they are first changed to the simplest forms. Thus, to show that $2x+3y=4$ and $6x+9y=15$ are inconsistent, divide each term of the second equation by 3. This gives $2x+3y=5$ which is contradictory to the equation $2x+3y=4$.

Exercises

Show that the following pairs of equations are inconsistent:

1. $x+y=5$
$2x+2y=14.$

2. $x-y=3$
$3x-3y=5.$

3. $4x+6y=24$
$6x+9y=30.$

4. $2x-y=7$
$3y-6x=10.$

5. $6x-4y=8$
$4y-6x=10.$

6. $3x-y=-4$
$9x-3y=1.$

7. $6x+2y-3=0$
$2x+y-8=0.$

8. $3x+\frac{y}{2}=4$
$2x+\frac{y}{3}=5.$

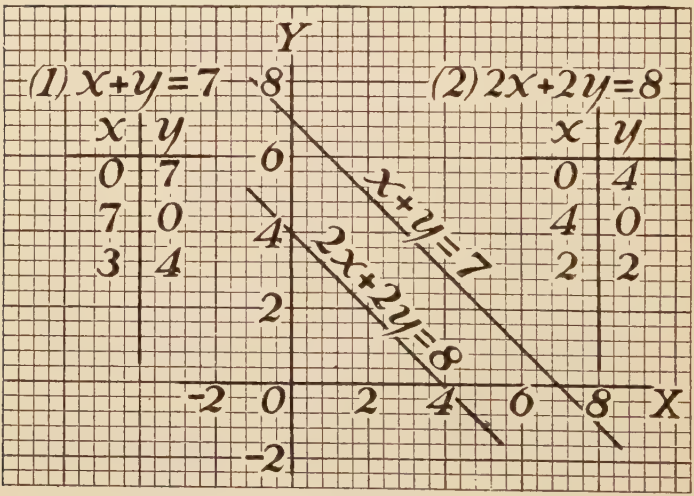


Fig. 62

160. Inconsistent equations represented graphically. We may show graphically that two inconsistent

equations cannot have a solution in common. For example, make the graphs of the inconsistent equations $x+y=7$ and $2x+2y=8$ (Fig. 62). You will see that the straight lines representing equations (1) and (2) are parallel. Hence, they cannot intersect, however far extended.

In general, the graphs of two inconsistent linear equations in two unknowns are two parallel straight lines.

Exercises

Show graphically that the equations in the exercises of §159 are inconsistent.

161. Equivalent equations. You have seen that a pair of linear equations in two unknowns may have *one* solution. In that case the two straight lines which represent the equations intersect, and the equations are *consistent.*

You have also seen that the two equations may have *no* solution in common. The graphs are then found to be parallel, and the equations are *inconsistent.*

There is still another possibility: Any solution of one of the two equations may be a solution of the other. In that case the equations are said to be *equivalent.*

Thus, the equations $2x - y = 10$
and $6y - 12x = -60$

reduce to one and the same equation by dividing the second equation by -6. Hence, a solution of one must be a solution of the other, and the graphs must be identical, *i.e.*, the same straight line.

Exercises

Show that the following equations are equivalent:

1. $2x+3y=12$
   $6x+9y=36.$

2. $2x-5y=4$
   $10y=4x-8.$

3. $x+\frac{3}{4}y=2$
   $\frac{x}{3}+\frac{y}{4}=\frac{2}{3}.$

4. $2x+4y=9$
   $x+2y=4\frac{1}{2}.$

5. $6y=3x-2$
   $15x=30y+10.$

6. $x=10-y$
   $2\frac{1}{2}y-25=-2\frac{1}{2}x.$

Principles of Logarithms

162. A law for finding the logarithm of a product. In multiplying numbers by means of logarithms the following principle is used: *The logarithm of a product is equal to the sum of the logarithms of the factors.* The truth of this law may be established as follows:

Let ab be the product of any two given numbers a and b.

You are to show that $\log ab = \log a + \log b$.

Proof: Let the logarithm of a be denoted by m,

i.e., let $\log a = m$. (1)

Similarly, let $\log b = n$. (2)

Then, according to the meaning of the term *logarithm*, equations (1) and (2) may be changed to

$$a = 10^m$$
$$\text{and } b = 10^n.$$

Multiplying the last two equations, you have

$$ab = 10^m \times 10^n = 10^{m+n}$$
$$\text{or } ab = 10^{m+n}$$

The last equation means that $m+n$ is the logarithm of ab, *i.e.*, that $\log ab=m+n$.

From equations (1) and (2) it follows that

$$\log ab=\log a+\log b.$$

163. A law for finding the logarithm of a quotient. The law is used to reduce the process of division to that of subtraction. It is as follows: *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor, i.e.,* $\log \frac{a}{b}=\log a-\log b.$

To prove this law, let $\frac{a}{b}$ be the quotient of two given numbers a and b.

$$\text{Let } \log a=m$$
$$\text{and } \log b=n.$$

Changing these equations into the exponential form, you have

$$a=10^m$$
$$\text{and } b=10^n.$$

Divide the first equation by the second. The result is

$$\frac{a}{b}=\frac{10^m}{10^n}=10^{m-n}$$

$$\text{or } \frac{a}{b}=10^{m-n}$$

which may be stated in the logarithmic form

$$\log \frac{a}{b}=m-n.$$

Substitute $\log a$ and $\log b$ for m and n.

$$\text{Then } \log \frac{a}{b}=\log a-\log b.$$

164. A law for finding the logarithm of a power. The following law enables you to find the value of a power by means of logarithms: *The logarithm of a power is equal to the exponent multiplied by the logarithm of the base, i.e.,* $\log a^p = p \log a$.

$$\begin{aligned} &\text{Let } \log a = m. \\ &\text{Then } a = 10^m \\ &\therefore a^p = (10^m)^p = 10^{mp} \\ &\text{or } a^p = 10^{pm}. \end{aligned}$$

In logarithmic notation this means that

$$\begin{aligned} &\log a^p = pm \\ &\text{or } \log a^p = p \log a. \end{aligned}$$

165. A law for finding roots by means of logarithms. The process of extracting roots by logarithms is exceedingly simple. The following theorem is used: *The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root,* i.e.,

$$\log \sqrt[n]{a} = \frac{\log a}{n}.$$

To prove this let $\log a = m$.

$$\begin{aligned} &\text{Then } a = 10^m. \\ &\therefore a^{\frac{1}{n}} = (10^m)^{\frac{1}{n}} = 10^{\frac{m}{n}}. \\ &\therefore \log a^{\frac{1}{n}} = \frac{m}{n} \\ &\text{or } \log \sqrt[n]{a} = \frac{\log a}{n}. \end{aligned}$$

Determinants

166. A formula for solving two linear equations in two unknowns. The equation $ax + by = c$ is a *general*

linear equation in two unknowns. Any special equation is obtained from $ax+by=c$ by assigning values to a, b, and c. Thus $2x-4y=3$ is a special case, where $a=2$, $b=-4$, and $c=3$. By solving the system

$$\begin{aligned} ax+by&=c \\ dx+ey&=f \end{aligned}$$

it is possible to derive a formula for finding the solution of any given special system.

Solution: $ax+by=c$ (1)

$dx+ey=f.$ (2)

Multiply (1) by d and (2) by a. Then

$$dax+dby=cd \quad (3)$$
$$dax+eay=af. \quad (4)$$

Subtract (4) from (3). This gives

$$dby-eay=cd-af.$$

Combining terms you have

$$(db-ea)y=cd-af.$$

Divide both sides of the equation by $db-ea$.

Then $$y=\frac{cd-af}{db-ea}.$$

To find x, eliminate y from equations (1) and (2): Multiplying (1) by e and (2) by b you have

$$\begin{aligned} eax+eby&=ec \\ bdx+bey&=bf. \end{aligned}$$

Subtract and combine terms. Then

$$(ea-bd)x=ec-bf.$$

$$\therefore x=\frac{ec-bf}{ea-bd}.$$

$$\therefore (x,\ y)=\left(\frac{ec-bf}{ea-bd},\ \frac{cd-af}{db-ea}\right).$$

Before you use this formula it is desirable to change it into a more convenient form (§168).

167. What is meant by a determinant. The binomials $ec-bf$, $ea-bd$, $cd-af$, $db-ea$ are called *determinants.* Each is the difference of the products of two factors. Usually the determinants are written in a different form. If in the equations

$$\begin{aligned} ax+by&=c \\ dx+ey&=f \end{aligned}$$

the x, y, and the signs of equality are omitted you have the following arrangement of the remaining literal numbers:

$$\begin{matrix} a, & b, & c \\ d, & e, & f. \end{matrix}$$

If they are written in the form of squares, as

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} \qquad \begin{vmatrix} a & c \\ d & f \end{vmatrix} \qquad \begin{vmatrix} c & b \\ f & e \end{vmatrix}$$

the following meaning is assigned to them:

Each square denotes the difference of the two cross products of the numbers contained in it. Thus, in the first square the cross products are ae and bd, and the square means $ae-bd$, *i.e.*,

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = ae-bd.$$

Similarly, $\begin{vmatrix} a & c \\ d & f \end{vmatrix} = af-dc.$

and $\begin{vmatrix} c & b \\ f & e \end{vmatrix} = ce-bf.$

The preceding discussion may now be summarized as follows:

1. Symbols like $\begin{vmatrix} a & b \\ d & e \end{vmatrix}$ are called *determinants.*

2. $\begin{vmatrix} a & b \\ d & e \end{vmatrix}$ means $ae-bd$.

3. To obtain the second form, $ae-bd$, from the first form $\begin{vmatrix} a & b \\ d & e \end{vmatrix}$ multiply the number in the upper left-hand corner by the number in the lower right-hand corner, and subtract the product of the two remaining numbers.

Exercises

Change the following determinants into the difference of two products and find the value of each:

1. $\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}$

Solution: $\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} = 4\times5-3\times2=20-6=14.$

2. $\begin{vmatrix} 2 & 5 \\ 6 & 24 \end{vmatrix}$ **5.** $\begin{vmatrix} -5 & 4 \\ 2 & 6 \end{vmatrix}$ **8.** $\begin{vmatrix} 2 & -2 \\ 3 & 5 \end{vmatrix}$

3. $\begin{vmatrix} 5 & 3 \\ 7 & 2 \end{vmatrix}$ **6.** $\begin{vmatrix} 3 & 2 \\ 7 & -8 \end{vmatrix}$ **9.** $\begin{vmatrix} 6 & 2 \\ -4 & -3 \end{vmatrix}$

4. $\begin{vmatrix} 6 & 0 \\ 3 & 4 \end{vmatrix}$ **7.** $\begin{vmatrix} 6 & -2 \\ 3 & 4 \end{vmatrix}$ **10.** $\begin{vmatrix} -3 & -5 \\ 2 & 6 \end{vmatrix}$

168. How to use determinants in solving equations. You have seen that the solution of the system

$$\begin{matrix} ax+by=c \\ dx+ey=f \end{matrix} \quad \text{is} \quad \begin{cases} x=\dfrac{ce-bf}{ae-bd} \\ y=\dfrac{af-cd}{ae-bd}. \end{cases}$$

Changing the determinants into the square form explained in §167 you have

$$x=\frac{ce-bf}{ae-bd}=\frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

$$y=\frac{af-cd}{ae-bd}=\frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}.$$

These results suggest the following method of solving the system

$$\begin{matrix} ax+by=c \\ dx+ey=f: \end{matrix}$$

1. Note that the denominators for x and y are the same; namely, the determinant formed by the coefficients of x and y. Hence, write the denominators first, as $x=\dfrac{\quad}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$, $y=\dfrac{\quad}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$

2. From the denominator form the numerator for x by replacing the coefficients of x by the numbers on the right sides of the equations.

This gives the numerator for x: $\begin{vmatrix} c & b \\ f & e \end{vmatrix}$

3. Form the numerator for y from the denominator by replacing the coefficients of y by the numbers on the right sides of the equations.

This gives the numerator for y: $\begin{vmatrix} a & c \\ d & f \end{vmatrix}$

Exercises

Solve the following systems by means of determinants:

1. $3x-13y=41$
$8x+11y=18.$

Solution: The denominator for x and y is $\begin{vmatrix} 3 & -13 \\ 8 & 11 \end{vmatrix}$.

The numerator for x is $\begin{vmatrix} 41 & -13 \\ 18 & 11 \end{vmatrix}$.

The numerator for y is $\begin{vmatrix} 3 & 41 \\ 8 & 18 \end{vmatrix}$.

$$\therefore x=\frac{\begin{vmatrix} 41 & -13 \\ 18 & 11 \end{vmatrix}}{\begin{vmatrix} 3 & -13 \\ 8 & 11 \end{vmatrix}}=\frac{41\times 11-(-13)(18)}{3\times 11-(-13)(8)}$$

$$=\frac{451+234}{33+104}=\frac{685}{137}=5.$$

$$y=\frac{\begin{vmatrix} 3 & 41 \\ 8 & 18 \end{vmatrix}}{137}=\frac{54-328}{137}=\frac{-274}{137}=-2.$$

$$\therefore (x,\ y)=(5,\ -2).$$

2. $5x+\ y=11$
$3x+2y=\ 1.$

3. $7x-3y=-18$
$4x-5y=-\ 7.$

4. $2x+7y=52$
$3x-5y=16.$

5. $7x+3y-23=0$
$5x+3y-19=0.$

6. $x+2y=0$
$x-\ y=2.$

7. $x+2.5y=150$
$2x-\ 4y=-80.$

8. $\frac{3x}{4}-\frac{5y}{6}=1$
$\frac{5x}{6}-\frac{3y}{4}=2.$

9. $\frac{x}{a}+\frac{y}{b}=1$
$\frac{x}{c}+\frac{y}{d}=1.$

FORMULAS

Direct variation: $y = cx.$

$$\frac{y_1}{y_2} = \frac{x_1}{x_2}.$$

Inverse variation: $y = \frac{c}{x}.$

$$xy = c.$$

$$\frac{y_1}{y_2} = \frac{x_2}{x_1}.$$

Arithmetical progression: $l = a + (n-1)d.$

$$s = \frac{n}{2}(a+l).$$

Geometric progression: $l = ar^{n-1}.$

$$s = \frac{a - ar^n}{1-r}.$$

Infinite geometric progression:

$$s = \frac{a}{1-r}, \text{ if } r < 1.$$

Remainder theorem: If $f(x)$ is divided by $x-a$, the remainder is $f(a)$.

Trigonometric ratios: Let c denote the hypotenuse of a right triangle, a the side opposite angle A, and b the side adjacent to angle A.

$$\text{Then: } \sin A = \frac{a}{c}.$$

$$\cos A = \frac{b}{c}.$$

$$\tan A = \frac{a}{b}.$$

$$\sin^2 A + \cos^2 A = 1.$$

$$\tan A = \frac{\sin A}{\cos A}.$$

Theorem of Pythagoras:

$$a^2 + b^2 = c^2.$$

Factoring:

$$x^2 - y^2 = (x+y)(x-y).$$

$$x^3 - y^3 = (x-y)(x^2+xy+y^2).$$

$$x^3 + y^3 = (x+y)(x^2-xy+y^2).$$

$$x^2 \pm 2xy + y^2 = (x \pm y)^2.$$

$$ax + ay + az = a(x+y+z).$$

Exponents:

$$a^m \cdot a^n = a^{m+n}.$$

$$\frac{a^m}{a^n} = a^{m-n}.$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

$$(a^m)^n = a^{mn}.$$

$$a^0 = 1.$$

$$a^{-m} = \frac{1}{a^m}.$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Binomial theorem:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \ . \ . \ .$$

$$t_r = \frac{n(n-1)\ . \ . \ . \ (n-r+2)}{1\cdot 2\ . \ . \ . \ (r-1)}a^{n-r+1}b^{r-1}$$

Logarithms:

$$\log ab = \log a + \log b.$$

$$\log \frac{a}{b} = \log a - \log b.$$

$$\log a^m = m \log a.$$

$$\log \sqrt[n]{a} = \frac{\log a}{n}.$$

Systems of equations in two unknowns:

$$ax + by = c$$
$$dx + ey = f.$$

$$x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}. \quad y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}.$$

Quadratic equations in one unknown:

$$ax^2 + bx + c = 0.$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$x_1 + x_2 = -\frac{b}{a}.$$

$$x_1 \cdot x_2 = \frac{c}{a}.$$

Quadratic equations in two unknowns:

Parabola: $y=ax^2+bx+c.$

$x=ay^2+by+c.$

Circle: $x^2+y^2=r^2.$

Ellipse: $ax^2+by^2=c.$

Hyperbola: $ax^2-by^2=c.$

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